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Dilaton deformations in closed string field theory

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Abstract: The dilaton theorem implies that the contribution to the dilaton potential from cubic interactions of all levels must be cancelled by the elementary quartic self-coupling of dilatons. We use this expectation to test the quartic structure of closed string field theory and to study the rules for level expansion. We explain how to use the results of Moeller to compute quartic interactions of states that, just like the dilaton, are neither primary nor have a simple ghost dependence. Our analysis of cancellations is made richer by discussing simultaneous dilaton and marginal deformations. We find evidence for two facts: as the level is increased quartic interactions become suppressed and closed string field theory may be able to describe arbitrarily large dilaton deformations.

Keywords: String Field Theory, Bosonic Strings.

Contents

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1. Introduction and summary

Level-expansion computations in open string field theory have been a useful tool in the study of open string tachyon condensation [1]. The early attempts to compute the potential for the (bulk) closed string tachyon of bosonic strings [2, 3] were done before level expansion was understood and the results were inconclusive. Clearer results were obtained recently in the computation of potentials for the twisted tachyons [4, 5] that live on orbifold cones. A workable level expansion scheme requires that finite number of couplings be considered at each computation stage. Since closed string field theory is nonpolynomial it is not obvious that level expansion works. If a class of closed-string computations can be done in level expansion, it is then necessary to compute higher-order couplings efficiently. The results of Moeller [6] make this possible for the case of four-point couplings. Moeller has provided the Riemann-surface data necessary to compute arbitrary couplings of four string fields: a concrete description of the subspace $\mathcal{V}_{0,4} \subset \mathcal{M}_{0,4}$ of the moduli space of four-punctured spheres and the local coordinates around the four punctures for every punctured sphere in $\mathcal{V}_{0,4}$.

In a previous paper [7] we considered marginal deformations in closed string field theory. The marginal parameter, called a, was that associated with the dimension-zero primary operator $c\bar{c}\partial X\bar{\partial}X$. When the coordinate X lives on a circle the operator induces a change of radius. The operator is marginal even when the coordinate X is noncompact, but adding it to the action does not change the correlators of the conformal theory. We used this marginal operator to test the quartic structure of closed string field theory and the feasibility of level expansion. We checked the vanishing of the effective potential for a. In the level expansion the quartic terms generated by the cubic interactions (to all levels) must be cancelled by the elementary quartic interaction of four marginal operators. We confirmed this prediction, thus giving evidence that the sign, normalization, and region of integration $\mathcal{V}_{0,4}$ for the quartic vertex are all correct. This was the first calculation of an elementary quartic amplitude for which there was an expectation that could be checked. We also extended the calculation to the case of the four marginal operators associated with two space coordinates.

In this paper we consider a nontrivial extension of the above results. We study the potential for the zero-momentum dilaton, the field d associated with the operator $\frac{1}{2}(c\partial^2 c - \bar{c}\bar{\partial}^2\bar{c})$. Complications arise because this operator is not marginal: it has dimension zero but it is not primary since it fails to be annihilated by L_1 and by $\bar{L}_1$. The dilaton theorem [8] states that a shift in the expectation value of d corresponds to a change in the string coupling constant. Around the flat spacetime background there is no potential for the dilaton, so it behaves like a marginal field. Therefore, a prediction similar to that for the field a exists: the quartic terms d^4 induced by the cubic interactions (to all levels) must be cancelled by the elementary quartic interaction of four dilatons. We will verify this prediction.

In closed string field theory all quartic terms that have been computed to date involve states that are primary and have $c\bar{c}$ ghost dependence. These states are off-shell only because their dimension is not zero. The computations of elementary four-string couplings in this paper involve dilaton states, which are nonprimary states with non-standard ghost dependence. The antighost insertions of the four-string interaction become quite nontrivial: they are not of the form $b_{-1}\bar{b}_{-1}$ acting on the moving puncture. The steps that must be taken using Moeller's data to compute such general four-point interactions are explained in detail in section 3.

Our analysis focuses on the two-dimensional space of deformations generated by a and d. The simultaneous marginality implies that the cancellation between cubic and quartic contributions in the effective potential holds for a^4, d^4 *and* a^2d^2. The computations of the elementary quartic amplitudes d^4 and a^2d^2 are done in section 4. The success in testing these cancellations provides evidence that the setup in section 3 works correctly and that the data of Moeller captures sophisticated information about the local coordinates on the punctures of a class of spheres that enter into the string vertex. Since the cancellations must happen for any four-string vertex that is consistent with the chosen three-string vertex, the amplitudes that are integrated over $\mathcal{V}_{0,4}$ are in fact total derivatives. In this way the quartic couplings only depend on the boundary $\partial\mathcal{V}_{0,4}$, which is indeed determined by the three string vertex by the condition of gauge invariance.

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Our interest on computations involving the dilaton arises from additional reasons. At some degree of accuracy most closed string theory computations involve the dilaton. Condensation of the dilaton plays a role in the Hagedorn transition: the coupling of the dilaton to nearly relevant states suggests that this transition is first order and occurs below the Hagedorn temperature [9]. The dilaton must certainly condense in the decay of orbifold cones [10−12]. Finally, we expect the dilaton to be relevant to the computation of the bulk tachyon potential.

In section 2 we focus on computations that involve only the quadratic and cubic terms in the closed string field theory action. We begin by calculating the effective potential $\mathbb{V}(a,d)$ obtained by integrating out the tachyon field. We find that the domain of definition of the marginal direction a is bounded, as it was for the Wilson line parameter in open string field theory [13] but interestingly, at least to this level, dilaton deformations are not bounded. This suggests the attractive possibility that closed string field theory may be able to describe arbitrarily large dilaton deformations (additional evidence is discussed in section 5). We then compute contributions to the quartic terms in $\mathbb{V}(a,d)$ from closed string states of level less than or equal to six. This gives us enough data to perform a rough extrapolation to infinite level. For the terms quartic on the dilaton in $\mathbb{V}(a,d)$ we push the calculation to higher level by exploiting the factorization of correlators. Intriguingly, the closed string computation is related to a computation in the *quantum* gauge-fixed open string action.

In section 3 we discuss the computation of general quartic elementary interactions, paying particular attention to the antighost insertions and collecting a series of results that allow the straightforward calculation of such interactions. In section 4 we perform the computations of the quartic couplings a^2d^2 and d^4 needed for $\mathbb{V}(a,d)$.

Section 5 is our concluding section. We analyze in detail the expected cancellations using an infinite-level extrapolation of the cubic contributions. We discuss a definition of level suitable for quartic interactions and find evidence that as the level is increased quartic interactions are suppressed, just as it happens for cubic interactions. Finally, we state some open problems and suggest possible directions for investigation.

2. Dilaton and marginal field potential from cubic interactions

With $\alpha' = 2$ the closed string field potential V is given by [14, 15]

$$\kappa^2 V = \frac{1}{2}\langle\Psi|c_0^- Q|\Psi\rangle + \frac{1}{3!}\{\Psi,\Psi,\Psi\} + \frac{1}{4!}\{\Psi,\Psi,\Psi,\Psi\} + \cdots . \qquad (2.1)$$

A state $|\Psi\rangle$ in closed string spectrum is a ghost number two state that satisfies $(L_0 - \bar{L}_0)|\Psi\rangle = 0$ and $(b_0 - \bar{b}_0)|\Psi\rangle = 0$. We fix the gauge invariance of the theory using the Siegel gauge $(b_0 + \bar{b}_0)|\Psi\rangle = 0$. The level ℓ of a state is defined by $\ell = L_0 + \bar{L}_0 + 2$. The closed string tachyon has level zero and fields corresponding to marginal directions have level two. We have $c_0^{\pm} = \frac{1}{2}(c_0 \pm \bar{c}_0)$ and the BRST operator is $Q = c_0 L_0 + \bar{c}_0\bar{L}_0 + \ldots$, where the dots denote terms independent of c_0 and of $\bar{c}_0$. We normalize correlators using $\langle 0|c_{-1}\bar{c}_{-1}c_0^- c_0^+ c_1\bar{c}_1|0\rangle = 1$ and note that

$$\langle c(z_1)c(z_2)c(z_3)\,\bar{c}(\bar{w}_1)\bar{c}(\bar{w}_2)\bar{c}(\bar{w}_3)\rangle = -2\langle c(z_1)c(z_2)c(z_3)\rangle_o \cdot \langle \bar{c}(\bar{w}_1)\bar{c}(\bar{w}_2)\bar{c}(\bar{w}_3)\rangle_o\,, \qquad (2.2)$$

where $\langle c(z_1)c(z_2)c(z_3)\rangle_o = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$ is the conventional open string field theory correlator. The cubic amplitude for three zero-momentum closed string tachyons is

$$\langle c_1\bar{c}_1, c_1\bar{c}_1, c_1\bar{c}_1\rangle = 2\mathcal{R}^6\,, \qquad \text{with} \qquad \mathcal{R} \equiv \frac{1}{\rho} = \frac{3\sqrt{3}}{4} \simeq 1.2990\,. \tag{2.3}$$

Here ρ is the (common) mapping radius of the disks that define the three-string vertex.

2.1 Direct closed string computation

In our previous work [7] we calculated the potential for the marginal direction a associated with the state $\alpha_{-1}\bar{\alpha}_{-1}\, c_1\bar{c}_1|0\rangle$. This time we want to include the zero-momentum ghost dilaton d associated with the state $(c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle$. This state does not fit the strict definition of a marginal state: it has dimension $(0,0)$ but it is not a Virasoro primary. Nevertheless, the dilaton theorem indicates that this field has a vanishing potential. Including the tachyon and the massless fields the string field is

$$|\Psi_0\rangle = t\, c_1\bar{c}_1|0\rangle + a\,\alpha_{-1}\bar{\alpha}_{-1}\, c_1\bar{c}_1|0\rangle + d(c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle\,. \tag{2.4}$$

The subscript on the string field (and in the potentials below) denotes the level of the highest level *massive* field included. The ghost structure of the zero-momentum dilaton implies that the cubic vertex cannot couple *one* dilaton to any quadratic combination of t and a. Moreover, the cubic vertex cannot couple three dilatons, nor it can couple two dilatons to an a. The only possible three-point coupling that involves a dilaton is td^2. The corresponding term in the potential $\kappa^2 V$ is

$$\frac{1}{3!}\cdot 3\cdot(-2)\cdot t\, d^2\langle c_1\bar{c}_1\,, c_{-1}c_1\,, \bar{c}_{-1}\bar{c}_1\rangle\,, \tag{2.5}$$

where the factor of 3 arises from three possible ways to choose the puncture for the tachyon and the (-2) from the two cross-terms that contribute from the dilaton insertions. Using the factorization property (2.2), we find

$$+2\,t\,d^2\langle c_1\,, c_{-1}c_1\,, 1\rangle_o\,\langle \bar{c}_1\,, 1\,, \bar{c}_{-1}\bar{c}_1\rangle_o = 2\,t\,d^2\left(-\frac{8}{27}\mathcal{R}^3\right)\left(\frac{8}{27}\mathcal{R}^3\right) = -\frac{27}{32}\,t\,d^2\,. \tag{2.6}$$

To evaluate the open string amplitudes we have used the conservation law for the c_{-1} ghost oscillator (see [16, eq. (4.6)]). The potential computed without including vertices higher than cubic is:

$$\kappa^2 V_{(0)} = -t^2 + \frac{6561}{4096}\,t^3 + \frac{27}{16}\,ta^2 - \frac{27}{32}\,td^2\,. \tag{2.7}$$

To find the effective potential $V(a,d)$ we solve for t as a function of a and d. One readily finds:

$$t^{V/M} = \frac{8}{19683}\left(512 \pm \sqrt{2}\sqrt{131072 - 1062882\,a^2 + 531441\,d^2}\,\right). \tag{2.8}$$

There are two branches: the vacuum (V) branch and the marginal (M) branch. In the vacuum branch the tachyon has finite expectation value when a and d vanish — the expectation value corresponding to the stationary point of the cubic potential. In the marginal branch the tachyon expectation value vanishes when a and d vanish.

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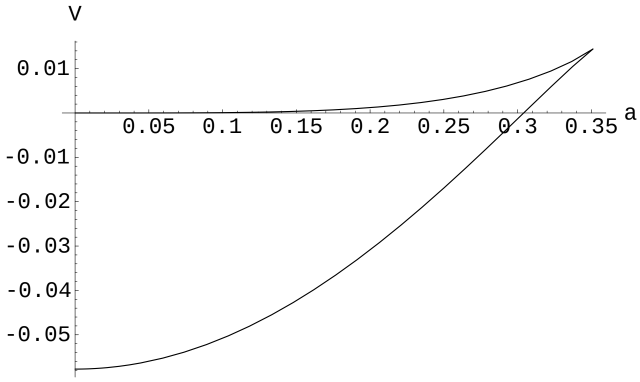

Figure 1: The effective potential for the radius deformation in the marginal branch (top) and in the tachyon branch (bottom).

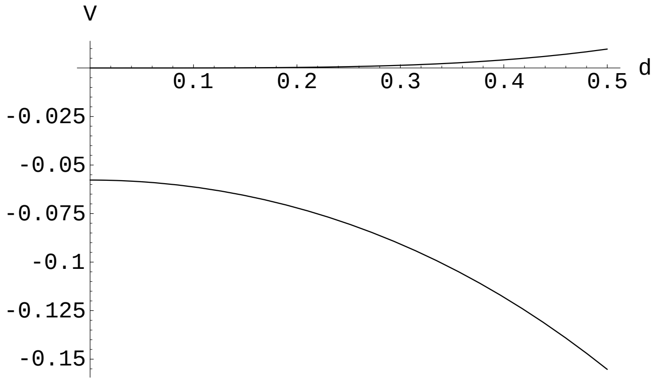

Figure 2: The effective potential for the dilaton in the marginal branch (top), and in the vacuum branch (bottom).

It follows from (2.8) that the effective potential is well defined as long as

$$1062882\, a^2 \ - \ 531441\, d^2 \leq 131072\,. \tag{2.9}$$

This implies that for $d = 0$ we must have $|a| \leq \frac{256}{729} \simeq 0.3512$. For $a = 0$, however, there is no constraint on the magnitude of d! This is a consequence of the minus sign with which the term $t\, d^2$ appears in the potential (2.7); the tachyon couples with *opposite* signs to a^2 and d^2. This is our first indication that closed string field theory is able to describe arbitrarily large dilaton deformations (in section 5 we cite further evidence to this effect).

In figure 1 we show the effective potential in the a-direction ($d = 0$) in both the marginal and tachyon branches. The qualitative features of this potential match those of the potential for the marginal deformation $c\partial X$ in open string field theory [13]: the marginal branch is reasonably flat and the two branches meet at the maximum possible value for a. In figure 2 we show the effective potential in the d-direction ($a = 0$) in both the marginal and tachyon branches. The marginal branch is roughly flat and the vacuum branch curves downward; the two branches do not meet. a and zero d).

Since the status of the stationary point in the cubic tachyon potential is still unclear, we focus henceforth on the marginal branch. We aim to use the quadratic and cubic terms in the string field theory to calculate the terms in $V(a,d)$ that are quartic in a and d, the leading terms for small a and d. We then want to show that these terms are cancelled by elementary quartic interactions. In order to calculate quartic terms, the t^3 interaction in (2.7) is not needed: the tachyon t is at least quadratic in a and d, so this interaction would contribute terms of order six in the marginal fields. Solving for the tachyon t as a function of a and d and substituting back in the potential, the quartic terms $\mathbb{V}_{(0)}$ in our calculation are:

$$\kappa^2 \mathbb{V}_{(0)} = \frac{3^6}{2^{10}} \left(a^4 - a^2 d^2 + \frac{d^4}{4} \right) \simeq 0.7119 \; a^4 - 0.7119 \; a^2 d^2 + 0.1780 \, d^4 \,. \tag{2.10}$$

We now turn to the computation to higher level, still using only the cubic vertex of the theory. In order to generate the required string field we note that the states are built with oscillators $\alpha_{n\leq -1}, \bar{\alpha}_{n\leq -1}$ of the coordinate X, Virasoro operators $L'_{m\leq -2}, \bar{L}'_{m\leq -2}$ corresponding to the remaining coordinates (thus $c = 25$), and ghost/antighost oscillators. We can list such fields systematically using the generating function:

$$\begin{aligned} f(x,\bar{x},y,\bar{y}) = & \prod_{n=1}^{\infty} \frac{1}{1-\alpha_{-n}x^n} \frac{1}{1-\bar{\alpha}_{-n}\bar{x}^n} \prod_{m=2}^{\infty} \frac{1}{1-L'_{-m}x^m} \frac{1}{1-\bar{L}'_{-m}\bar{x}^m} \cdot \\ & \cdot \prod_{\substack{k=-1 \\ k\neq 0}}^{\infty} (1+c_{-k}x^k y)(1+\bar{c}_{-k}\bar{x}^k\bar{y}) \prod_{l=2}^{\infty} (1+b_{-l}x^l y^{-1})\,(1+\bar{b}_{-l}\bar{x}^l \bar{y}^{-1}) \,. \end{aligned} \tag{2.11}$$

A term of the form $x^n \bar{x}^{\bar{n}} y^m \bar{y}^{\bar{m}}$ corresponds to a state with $(L_0, \bar{L}_0) = (n, \bar{n})$ and ghost numbers $(G, \bar{G}) = (m, \bar{m})$. Since the string field must have total ghost number two we require $m + \bar{m} = 2$. A massive field M is relevant to our calculation if it has a coupling Ma^2, or Md^2, or Mad, or any combination of them. If all three couplings vanish we can set M to zero consistently to this order. We readily see the following rules also apply:

- A field with $(G, \bar{G}) = (1,1)$ can couple only to a^2 and to d^2. Such field must have an even number of α's and an even number of $\bar{\alpha}$'s.

- A field with $(G, \bar{G}) = (0,2)$ or $(2,0)$ can couple only to ad. Such field must have an odd number of α's and an odd number of $\bar{\alpha}$'s.

- A field with $(G, \bar{G}) = (-1,3)$ or $(3,-1)$ can couple only to d^2. Such field must have an even number of α's and an even number of $\bar{\alpha}$'s.

At level $\ell = 4$ we have $L_0 = \bar{L}_0 = 1$. The coefficients of $(x\bar{x}y\bar{y}, x\bar{x}y^2, x\bar{x}\bar{y}^2)$ give all possible terms in the string field. With the above rules the set is reduced to

$$\begin{aligned} |\Psi_4\rangle = \; & f_1\, c_{-1}\bar{c}_{-1} + f_2\, L'_{-2}\bar{L}'_{-2} c_1 \bar{c}_1 + \left(f_3\, L'_{-2} c_1 \bar{c}_{-1} + f_4\, \bar{L}'_{-2} c_{-1} \bar{c}_1 \right) + \\ & + r_1\, \alpha_{-1}^2 \bar{\alpha}_{-1}^2 c_1 \bar{c}_1 + \left(r_2\, \alpha_{-1}^2 c_1 \bar{c}_{-1} + r_3\, \bar{\alpha}_{-1}^2 c_{-1} \bar{c}_1 \right) + \\ & + r_4\, \alpha_{-1}^2 \bar{L}'_{-2} c_1 \bar{c}_1 + r_5\, L'_{-2} \bar{\alpha}_{-1}^2 c_1 \bar{c}_1 + \left(r_6\, \alpha_{-1}\bar{\alpha}_{-1} c_{-1} c_1 + r_7\, \alpha_{-1}\bar{\alpha}_{-1} \bar{c}_{-1} \bar{c}_1 \right) . \end{aligned} \tag{2.12}$$

The corresponding terms in the potential $V_{(4)}$ are given in appendix A. Eliminating all massive fields through their equations of motion we obtain

$$\begin{aligned}\kappa^2 V_{(4)} &= -\frac{19321}{46656}\,a^4 + \frac{1619}{15552}\,a^2 d^2 - \frac{6241}{186624}\,d^4 \\ &\simeq -0.4141\,a^4 + 0.1041\,a^2 d^2 - 0.0334\,d^4\,. \end{aligned} \tag{2.13}$$

To get the total contribution up to level four we add the above to the result in (2.10):

$$\begin{aligned}\kappa^2 \mathbb{V}_{(4)} &= \frac{222305}{746496}\,a^4 - \frac{151243}{248832}\,a^2 d^2 + \frac{431585}{2985984}\,d^4 \\ &\simeq 0.2978\,a^4 - 0.6078\,a^2 d^2 + 0.1445\,d^4\,. \end{aligned} \tag{2.14}$$

We have computed the contribution from level six string fields. The states that contribute as well as the potential are given in appendix A. Eliminating out the massive fields we find:

$$\begin{aligned}\kappa^2 V_{(6)} &= \frac{53824}{531441}\,a^2 d^2 - \frac{5000}{177147}\,d^4 \\ &\simeq 0.1013\,a^2 d^2 - 0.0282\,d^4\,. \end{aligned} \tag{2.15}$$

The full set of quartic terms to level six is obtained by adding the above to (2.14):

$$\begin{aligned}\kappa^2 \mathbb{V}_{(6)} &= \frac{222305}{746496}\,a^4 - \frac{275652665}{544195584}\,a^2 d^2 + \frac{84395155}{725594112}\,d^4 \\ &= 0.2978\,a^4 - 0.5065\,a^2 d^2 + 0.1163\,d^4\,. \end{aligned} \tag{2.16}$$

By comparing the successive approximations $\mathbb{V}_{(0)}$, $\mathbb{V}_{(4)}$, and $\mathbb{V}_{(6)}$ we note that the coefficients of a^4, a^2d^2, and d^4 all decrease (in magnitude) as we increase the level. Introducing the notation

$$\begin{aligned}\kappa^2 V_{(\ell)} &= c_{a^4}(\ell)\,a^4 + c_{a^2d^2}(\ell)\,a^2\,d^2 + c_{d^4}(\ell)\,d^4 \\ \kappa^2 \mathbb{V}_{(\ell)} &= \mathcal{C}_{a^4}(\ell)\,a^4 + \mathcal{C}_{a^2d^2}(\ell)\,a^2\,d^2 + \mathcal{C}_{d^4}(\ell)\,d^4 \end{aligned} \tag{2.17}$$

the information obtained so far is collected in table 1. The table shows both the contributions that arise from each level ℓ of the massive fields (the quantities $c_{\ldots}(\ell)$) and the total contributions up to that level (the quantities $\mathcal{C}_{\ldots}(\ell)$). Even at infinite level the total contributions to the quartic coefficients do not vanish. The infinite-level cubic calculations give a quartic potential that must be cancelled by the contributions from the elementary quartic interactions.

2.2 Contributions to d^4 calculated using OSFT

To extend the computations of the previous subsection to higher levels requires significant work. In [7], we showed how to obtain the contributions to a^4 in the closed string effective potential in terms of analogous contributions to the potential of the Wilson line parameter a_s in open string field theory. This comparison worked because, in the Siegel gauge, the closed string kinetic and cubic terms factorize into holomorphic and antiholomorphic correlators which feature in analogous open string computations.

ℓ	$c_{a^4}(\ell)$	$\mathcal{C}_{a^4}(\ell)$	$c_{a^2d^2}(\ell)$	$\mathcal{C}_{a^2d^2}(\ell)$	$c_{d^4}(\ell)$	$\mathcal{C}_{d^4}(\ell)$
0	0.7119	0.7119	−0.7119	−0.7119	0.1780	0.1780
4	−0.4141	0.2978	0.1041	−0.6078	−0.0334	0.1445
6	0	0.2978	0.1013	−0.5065	−0.0282	0.1163

Table 1: The coefficients of the quartic terms in the effective potential for a and d as a function of the level ℓ of the massive fields integrated out.

In this subsection we compute additional coefficients $c_{d^4}(2\ell)$ in the dilaton potential. The factorization property cuts down significantly the number of correlators that must be computed. As we show at the end of this subsection, the computation of the dilaton potential is at least formally related to a computation in the *quantum* gauge fixed open string field theory.

The three string vertex couples d^2 to massive fields of ghost number $(1,1)$, $(-1,3)$, and $(3,-1)$. We thus consider the massive closed string field of level 2ℓ:

$$|\Psi\rangle = \sum_{i,j} \psi_{ij}\, |\mathcal{O}_i\rangle \otimes |\overline{\mathcal{O}}_j\rangle + \sum_{\alpha,a} \left(\psi_{\alpha a}\, |\mathcal{O}_\alpha\rangle \otimes |\overline{\mathcal{O}}_a\rangle \,+\, \psi_{a\alpha}\, |\mathcal{O}_a\rangle \otimes |\overline{\mathcal{O}}_\alpha\rangle\right). \tag{2.18}$$

The holomorphic basis states $|\mathcal{O}_i\rangle, |\mathcal{O}_\alpha\rangle$, and $|\mathcal{O}_a\rangle$ are all level ℓ open string states in the Siegel gauge, with ghost numbers $1, -1$, and 3, respectively:

$$G(\mathcal{O}_i) = 1\,, \qquad G(\mathcal{O}_\alpha) = -1\,, \qquad G(\mathcal{O}_a) = 3\,. \tag{2.19}$$

The barred states are identical looking states built with antiholomorphic oscillators.

The computation of the quadratic and cubic terms in the closed string action is helped by the following definitions:

$$\begin{aligned} m_{ij} &\equiv \langle \mathcal{O}_i | c_0 | \mathcal{O}_j\rangle = m_{ji}\,, \qquad R_{\alpha a} \equiv \langle \mathcal{O}_\alpha | c_0 | \mathcal{O}_a\rangle = \langle \mathcal{O}_a | c_0 | \mathcal{O}_\alpha\rangle\,, \\ K_i &\equiv \langle \mathcal{O}_i, c_1c_{-1}, \mathbf{1}\rangle\,, \qquad P_a \equiv \langle \mathcal{O}_a, \mathbf{1}, \mathbf{1}\rangle\,, \qquad Q_\alpha \equiv \langle \mathcal{O}_\alpha, c_1c_{-1}, c_1c_{-1}\rangle\,. \end{aligned} \tag{2.20}$$

We now want to evaluate the potential

$$\kappa^2 V_{2\ell} = \frac{1}{2}\langle \Psi | c_0^- \, Q_B | \Psi\rangle + \frac{1}{2}\{\Psi, D, D\}\,, \tag{2.21}$$

where $|D\rangle = d\,(c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle$ and the state $|\Psi\rangle$ is given in (2.18). Using the factorization property of correlators and the above definitions we find

$$\begin{aligned} \kappa^2 V_{2\ell} = {} & (\ell-1)\, m_{ii'}\, m_{jj'}\, \psi_{ij}\, \psi_{i'j'} + 2(\ell-1) R_{\alpha b}\, R_{\beta a}\, \psi_{\alpha a}\, \psi_{b\beta} + \\ & + 2(-)^\ell\, K_i\, K_j\, \psi_{ij}\, d^2 \; - Q_\alpha\, P_a\, \psi_{\alpha a}\, d^2 - P_b\, Q_\beta\, \psi_{b\beta}\, d^2\,, \end{aligned} \tag{2.22}$$

where repeated indices are summed over. The factor $(-1)^\ell$ in the third term of the right hand side arises because $\langle \mathcal{O}_i, \mathbf{1}, c_1c_{-1}\rangle = \Omega_{\mathcal{O}_i}\langle \mathcal{O}_i, c_1c_{-1}, \mathbf{1}\rangle = (-1)^\ell K_i$. Solving the (linear) equations of motion for $\psi_{ij}, \psi_{a\alpha}$, and $\psi_{\alpha a}$ and substituting back into the potential one finds

$$\kappa^2 V_{2\ell} = c_{d^4}(2\ell)\, d^4\,, \qquad \text{with} \qquad c_{d^4}(2\ell) = -\frac{1}{\ell-1}\left[\left(K^T M^{-1} K\right)^2 + \frac{1}{2}\left(P^T R^{-1} Q\right)^2\right]. \tag{2.23}$$

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The first term in the bracket gives the contribution to the potential from the $(1,1)$ massive fields and the second term gives the contribution to the potential from the $(-1,3)$ and $(3,-1)$ massive fields.

Let us now compute c_{d^4} for levels ranging from zero to ten. We use the open string universal subspace with matter Virasoro operators of central charge $c = 26$. The twist property

$$\langle A, B, C\rangle = \Omega_A \Omega_B \Omega_C (-1)^{BC+1} \langle A, C, B\rangle \tag{2.24}$$

of the cubic open string vertex implies that for B Grassmann even $\langle A, B, B\rangle = -\Omega_A \langle A, B, B\rangle$. Consequently $\langle A, B, B\rangle$ vanishes for twist even A or, equivalently, for states A of even level. We deduce that the vectors P_a and Q_α vanish for states at even levels.

- At $\ell = 0$ the only massive state is the tachyon:

$$M = 1\,, \qquad K = \frac{3\sqrt{3}}{8}\,, \qquad \Rightarrow c_{d^4}(2) = \frac{729}{4096} \simeq 0.177979\,, \tag{2.25}$$

  which confirms our result in (2.10).

- At $\ell = 2$ there are two ghost number one states, $c_{-1}|0\rangle$ and $L_{-2}c_1|0\rangle$. We find

$$\begin{aligned} M \;&=\; \mathrm{diag}(-1,\ 13)\,, \qquad K = \left(\tfrac{\sqrt{3}}{24},\, -\tfrac{65}{24\sqrt{3}}\right), \qquad K^T M^{-1} K = \frac{79}{432}\,, \\ &\Rightarrow c_{d^4}(4) = -\frac{6241}{186624} \simeq -0.0334416\,, \end{aligned} \tag{2.26}$$

  which agrees with the result in (2.13).

- At $\ell = 3$ there are three ghost number one states: $c_{-2}|0\rangle$, $b_{-2}c_{-1}c_1|0\rangle$, and $L_{-3}c_1|0\rangle$; one ghost number minus one state $b_{-2}|0\rangle$ and one ghost number three state $c_{-2}c_{-1} \times c_1|0\rangle$. Using the order in which we listed the states the relevant matrices are:

$$\begin{aligned} M &= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -52 \end{pmatrix}, \qquad K = -\frac{5}{27}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \;\Rightarrow\; K^T M^{-1} K = -\frac{100}{729}\,, \\ R &= 1\,, \quad Q = -\frac{20}{27}\,, \qquad P = -\frac{10}{27}\,, \quad \Rightarrow \quad P^T R^{-1} Q = \frac{200}{729}\,. \end{aligned} \tag{2.27}$$

  The contribution to c_{d^4} is then

$$c_{d^4}(6) = -\frac{5000}{531441} - \frac{10000}{531441} = -\frac{5000}{177147} \simeq -0.0282251\,, \tag{2.28}$$

  which is the result obtained in (2.15).

- At level $\ell = 4$, $P = Q = 0$ as we argued before. There are six ghost number one states:

$$c_{-3}|0\rangle\,, \quad b_{-3}c_{-1}c_1|0\rangle, \quad b_{-2}c_{-2}c_1|0\rangle\,, \qquad L_{-2}c_{-1}|0\rangle\,, \qquad L_{-2}^2 c_1|0\rangle\,, \qquad L_{-4}c_1|0\rangle\,. \tag{2.29}$$

We find $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \ \mathrm{diag}(\,-1,\,-13\,,\,442\,,\,130\,)$ as well as

$$K^T = \left(-\frac{5}{27\sqrt{3}}\,, \frac{5}{9\sqrt{3}}\,, \ -\frac{343}{648\sqrt{3}}\,, -\frac{65}{216\sqrt{3}}\,, \frac{3523}{324\sqrt{3}}\,, \frac{65}{12\sqrt{3}} \right). \tag{2.30}$$

We find $K^T M^{-1} K = \frac{37}{396576}$, and therefore,

$$c_{d^4}(8) = -\frac{1369}{471817571328} \simeq -2.9 \times 10^{-9}\,. \tag{2.31}$$

This number is anomalously small. Clearly, the coefficients $c_{d^4}(\ell)$ do not settle into a regular pattern for small ℓ.

- At level $\ell = 5$, there are nine ghost number one states:

$$\begin{aligned}
&c_{-4}|0\rangle\,,\ b_{-4}c_{-1}c_1|0\rangle\,,\ b_{-3}c_{-2}c_1|0\rangle\,,\ b_{-2}c_{-3}c_1|0\rangle\,,\ L_{-2}c_{-2}|0\rangle\,,\ L_{-2}b_{-2}c_{-1}c_1|0\rangle\,,\\
&L_{-3}c_{-1}|0\rangle\,,\ L_{-3}L_{-2}c_1|0\rangle\,,\ L_{-5}c_1|0\rangle\,,
\end{aligned} \tag{2.32}$$

three ghost number minus one states, and three ghost number three states:

$$\begin{aligned}
\mathcal{O}_\alpha: &\qquad b_{-4}|0\rangle\,, \qquad b_{-3}b_{-2}c_1|0\rangle\,, \qquad b_{-2}L_{-2}|0\rangle,\\
\mathcal{O}_a: &\qquad c_{-4}c_{-1}c_1|0\rangle\,, \qquad c_{-3}c_{-2}c_1|0\rangle\,, \qquad c_{-2}c_{-1}c_1L_{-2}|0\rangle\,.
\end{aligned} \tag{2.33}$$

We then find:

$$\begin{aligned}
M &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -13 \\ -13 & 0 \end{pmatrix} \oplus \mathrm{diag}(52, -936, -260),\\
K^T &= \frac{1}{729}\Big(39,\ 156,\ 75,\ -50,\ 325,\ 650,\ 0,\ 0,\ 0\Big)\,,
\end{aligned} \tag{2.34}$$

and

$$R = \mathrm{diag}(1,\ 1,\ 13)\,, \qquad Q^T = \left(\frac{104}{243}, \frac{100}{243}, \frac{1300}{729}\right), \qquad P^T = \left(\frac{26}{243}, \frac{50}{729}, \frac{650}{729}\right). \tag{2.35}$$

Therefore, one obtains:

$$\begin{aligned}
K^T M^{-1} K &= -\frac{52168}{531441}, \qquad PR^{-1}Q = \frac{104336}{531441}\,,\\
c_{d^4}(10) &= c_{d^4}^{(1,1)}(10) + c_{d^4}^{(-1,3)}(10) = -\frac{680375056}{282429536481}(1+2)\\
&= -\frac{680375056}{94143178827} \simeq -0.007227\,.
\end{aligned} \tag{2.36}$$

The total contribution up to level $\ell = 10$ is thus:

$$\mathcal{C}_{d^4}(\ell = 10) = \frac{12156561955612607}{111441423077388288} \simeq 0.109085\,. \tag{2.37}$$

ℓ	$c_{d^4}(\ell)$	$\mathcal{C}_{d^4}(\ell)$
0	0.1780	0.1780
4	-0.0334	0.1445
6	-0.0282	0.1163
8	-3×10^{-9}	0.1163
10	-0.0072	0.1091

Table 2: The coefficients of the quartic terms in the effective potential for d as a function of the level ℓ of the massive fields integrated out.

Speculations on related open string computation. The above closed string computation requires holomorphic correlators that appear naturally in an open string computation. We consider the gauge-fixed open string field theory and include spacetime fields u and v of ghost numbers one and minus one:

$$|A\rangle = |0\rangle u + c_1 c_{-1}|0\rangle v\,. \tag{2.38}$$

The spacetime fields appear as coefficients of the ghost number zero and ghost number two states. Since the open string field is Grassmann odd (and the vacuum $|0\rangle$ is Grassmann even) the fields u and v should be Grassmann odd. Note that the corresponding states have $L_0 = 0$ so they may be viewed as marginals of unusual ghost number. The kinetic term that could couple u and v vanishes. In fact, ghost number conservation implies that terms coupling only u fields or v fields must vanish. We claim that some of the ingredients in the computation of the open string effective action for A are closely related to those of the dilaton effective potential.

To calculate the effective potential for A we consider the massive open string field

$$|\Phi\rangle = \sum_i |\mathcal{O}_i\rangle\phi_i + \sum_\alpha |\mathcal{O}_\alpha\rangle\phi_\alpha + \sum_a |\mathcal{O}_a\rangle\phi_a\,, \tag{2.39}$$

and compute the potential

$$g^2 V_\ell = \frac{1}{2}\langle\Phi|\,Q_B|\Phi\rangle + \langle\Phi, A, A\rangle\,. \tag{2.40}$$

In this computation the cubic term couples uv to massive fields, but since $u^2 = v^2 = 0$ no other couplings appear. In order to make the computation more interesting and have P_a and Q_α feature in the result, we assume $uv = -vu$ but, in a departure from the conventional interpretation, no longer take u^2 and v^2 to vanish (one may imagine that this is a computation in which the string fields are two-by-two matrices and we take $u = \sigma_1$ and $v = \sigma_2$). Evaluation of the above action then gives:

$$g^2 V_\ell = \frac{1}{2}(\ell-1)\phi_i m_{ii'}\phi_{i'} + (\ell-1)\phi_\alpha R_{\alpha a}\phi_a + (1+(-1)^\ell)K_i\phi_i(vu) + P_a\Phi_a u^2 + Q_\alpha\phi_\alpha v^2\,. \tag{2.41}$$

We see that all the matrices and vectors introduced to write the closed string effective action appear here. With the assumption that u^2 and v^2 are nonvanishing one can compute the effective potential by integrating out the massive fields. The result is

$$g^2 V_\ell(u,v) = -\frac{1}{\ell-1}\left(\frac{(1+(-1)^\ell)^2}{2}(K^T M^{-1} K) - (P^T R^{-1} Q)\right)(uv)^2 . \tag{2.42}$$

In the standard interpretation this potential vanishes simply because $(uv)^2 = 0$, with no computation necessary. In that sense marginality seems preserved. Still the formal resemblance of (2.42) to (2.23) is quite intriguing. The sum $\sum_\ell V_\ell$ does not seem to converge to zero, so we do not understand the significance of the contributions V_ℓ.

It may be of interest to examine, after integration of massive fields, the quantum gauge fixed action at nonzero momentum. Here quartic terms would survive. Any exact relation between the classical closed string field theory action for the dilaton and the quantum, effective, gauge-fixed open string action may be quite illuminating.

3. Setting up elementary quartic computations

In a very useful piece of work Moeller [6] calculated the quadratic differential that defines the local coordinates on the punctures of the four-punctured spheres that comprise the quartic string vertex. He also gave a concrete description of the region of integration $\mathcal{V}_{0,4}$. This information is all that is needed, in principle, to compute any four-string coupling. In this section we show how to use this information to set up the computation of quartic interactions where the anti-ghost insertions play a nontrivial role — this happens whenever the states do not have the simple $c\bar{c}$ ghost dependence. The results that we obtain make the computation of four-string couplings straightforward. The specific computations required in this paper will be done in the following section.

3.1 The quartic term in the string action

The description of four-string amplitudes uses the definition [14, 2]

$$\{\Psi_1,\Psi_2,\Psi_3,\Psi_4\} \equiv \frac{i}{2\pi}\int_{\mathcal{V}_{0,4}} d\lambda_1\wedge d\lambda_2 \,\langle\Sigma|\, b(v_{\lambda_1}) b(v_{\lambda_2})|\Psi_1\rangle|\Psi_2\rangle|\Psi_3\rangle|\Psi_4\rangle . \tag{3.1}$$

Here $\langle\Sigma|$ is the (operator formalism) surface state corresponding to a four-punctured sphere $\Sigma\in\mathcal{V}_{0,4}$. In addition, λ_1 and λ_2 are two real parameters that describe the moduli space and $b(v_{\lambda_i})$, with $i=1,2$, are antighost factors given by

$$b(v_{\lambda_i}) = \sum_{I=1}^{4}\sum_{m=-1}^{\infty}\left(B^I_{i,m} b^I_m + \overline{B^I_{i,m}}\,\bar{b}^I_m\right), \quad \text{with} \quad B^I_{i,m} = \oint \frac{dw}{2\pi i}\frac{1}{w^{m+2}}\frac{1}{h_I'}\frac{\partial h_I}{\partial\lambda_i} . \tag{3.2}$$

The functions $h_I(w;\lambda_1,\lambda_2)$, with $I=1,\ldots,4$, define maps from a local coordinate w into the sphere described with uniformizer z, and the primes denote derivatives with respect to w. An overline on an number denotes complex conjugation. We use a presentation of the moduli space in which $(\lambda_1,\lambda_2)=(x,y)$, where $\xi = x+iy$ is the position of the moving puncture. It is then possible to rewrite the relevant two-form as [2]

$$d\lambda_1\wedge d\lambda_2\, b(v_{\lambda_1})b(v_{\lambda_2}) = d\xi\wedge d\bar{\xi}\,\mathcal{B}\,\mathcal{B}^\star , \tag{3.3}$$

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where

$$\mathcal{B} = \sum_{I=1}^{4} \sum_{m=-1}^{\infty} \left(B_m^I \, b_m^J + \overline{C_m^I} \, \bar{b}_m^I \right) , \qquad \mathcal{B}^\star = \sum_{I=1}^{4} \sum_{m=-1}^{\infty} \left(C_m^I \, b_m^I + \overline{B_m^I} \, \bar{b}_m^I \right) , \tag{3.4}$$

and the coefficients B_m^I and C_m^I are given by

$$B_m^J = \oint \frac{dw}{2\pi i} \frac{1}{w^{m+2}} \frac{1}{h_J'} \frac{\partial h_J}{\partial \xi} , \qquad C_m^J = \oint \frac{dw}{2\pi i} \frac{1}{w^{m+2}} \frac{1}{h_J'} \frac{\partial h_J}{\partial \bar{\xi}} . \tag{3.5}$$

In (3.3) we have introduced a $\star$-conjugation. Acting on a number $\star$-conjugation is just complex conjugation. Acting on a product of ghost oscillators $\star$-conjugation reverses their order and turns holomorphic oscillators into antiholomorphic ones, and viceversa. Note that this rule defines an involution and is consistent with (3.4). Note also that in (3.5) the functions $h_I(w; \xi, \bar{\xi})$ are simply the functions that describe the local coordinates written in terms of the complex modulus ξ.

Using (3.3) and $d\xi \wedge d\bar{\xi} = -2i \, dx \wedge dy$ the multilinear function in (3.1) becomes

$$\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\} \equiv \frac{1}{\pi} \int_{\mathcal{V}_{0,4}} dx \wedge dy \, \langle \Sigma | \, \mathcal{B} \, \mathcal{B}^\star \, |\Psi_1\rangle |\Psi_2\rangle |\Psi_3\rangle |\Psi_4\rangle \, . \tag{3.6}$$

The quartic term in the string field potential (2.21) is then given by ($\alpha' = 2$)

$$\kappa^2 V = \frac{1}{4!} \{\Psi, \Psi, \Psi, \Psi\} = \frac{1}{4!} \{ \, \Psi^4 \, \} \, . \tag{3.7}$$

The maps from local coordinates to the uniformizing coordinate z on the four-punctured sphere take the form:

$$z = h_I(w; \xi, \bar{\xi}) = z_I(\xi, \bar{\xi}) + \rho_I(\xi, \bar{\xi}) \, w + \rho_I^2 \, \beta_I(\xi, \bar{\xi}) \, w^2 + \rho_I^3 \gamma_I(\xi, \bar{\xi}) \, w^3 + \rho_I^4 \delta_I(\xi, \bar{\xi}) \, w^4 + \cdots \, . \tag{3.8}$$

Here ρ_I is a real, positive, number called the mapping radius. For convenience, factors of the mapping radius have been included in the definition of the higher order coefficients β_I, γ_I, and δ_I. We choose the first three punctures to be at $z = 0$, $z = 1$ and $z = \xi$, so

$$z_1(\xi, \bar{\xi}) = 0 \, , \quad z_2(\xi, \bar{\xi}) = 1 \, , \qquad \text{and} \qquad z_3(\xi, \bar{\xi}) = \xi \, . \tag{3.9}$$

While we can use (3.8) for $I = 1, 2, 3$, the fourth puncture is placed at $z = \infty$ and one must use a coordinate $t = 1/z$ which vanishes at this point. We thus write

$$t = h_4(w; \xi, \bar{\xi}) = \rho_4(\xi, \bar{\xi}) \, w + \rho_4^2 \, \beta_4(\xi, \bar{\xi}) \, w^2 + \rho_4^3 \, \gamma_4(\xi, \bar{\xi}) \, w^3 + \rho_4^4 \delta_4(\xi \, , \bar{\xi}) \, w^4 + \cdots \, . \tag{3.10}$$

All operators inserted at the fourth puncture must be thought as inserted at $t = 0$. One must then map them to $z = 1/t$ in order to compute correlators using the global uniformizer z.

Equations (3.5) allow us to express the coefficients B_m^I and C_m^I in terms of the expansion coefficients for the coordinates. With mode number minus one (the lowest possible one), we find

$$B_{-1}^J = \frac{1}{\rho_3} \delta_{3J} \, , \qquad C_{-1}^J = 0 \, . \tag{3.11}$$

Note that at this level the antighost insertion is supported only on the moving puncture, and by our choice $z=\xi$, the insertion is holomorphic. Since our string fields are annihilated both by b_0 and $\bar{b}_0$, the coefficients B_0^I and C_0^I are not needed.

The ghost-dilaton state contains ghost oscillators of mode number minus one. The coefficients B_1^I and C_1^I are thus needed to compute four-point amplitudes that involve dilatons. We then find

$$B_1^I = \rho_I \partial\beta_I + \frac{1}{2}\rho_3\varepsilon_3\delta_{I3}\,, \quad C_1^I = \rho_I\bar{\partial}\beta_I\,, \quad \text{with} \quad \varepsilon_I \equiv 8\,\beta_I^2 - 6\gamma_I\,, \quad \partial \equiv \frac{\partial}{\partial\xi}\,, \quad \bar{\partial} \equiv \frac{\partial}{\partial\bar{\xi}}\,. \tag{3.12}$$

A similar calculation gives

$$B_2^I = \rho_I^2\partial(\gamma_I - \beta_I^2) + \rho_I^2(-4\delta_I - 2\varepsilon_I\beta_I + 8\beta_I^3)\delta_{3I}, \quad C_2^I = \rho_I^2\bar{\partial}(\gamma_I - \beta_I^2)\,. \tag{3.13}$$

The above results suffice to compute four point amplitudes with states that contain oscillators c_{-n} and $\bar{c}_{-n}$ with $n \leq 2$. Taking note of the vanishing coefficients, we see that for states in the Siegel gauge the antighost factor $\mathcal{B}$ is given by

$$\mathcal{B} = B_{-1}^3 b_{-1}^{(3)} + \sum_{I=1}^{4}\Big(B_1^I\, b_1^J + \overline{C_1^I}\,\bar{b}_1^I\Big) + \sum_{I=1}^{4}\Big(B_2^I\, b_2^J + \overline{C_2^I}\,\bar{b}_2^I\Big) + \cdots\,. \tag{3.14}$$

In order to proceed further we learn how to obtain the coordinate expansion coefficients β_I, γ_I, and δ_I from quadratic differentials.

3.2 Strebel differential and local coordinates

Consider a four-punctured sphere with uniformizer z. Place the first, second, and fourth punctures at $0, 1$, and ∞, respectively, and let the third puncture be placed at $z=\xi$. The collection of four-punctured spheres that comprise the moduli space $\mathcal{V}_{0,4}$ can be described as the region of the complex z-plane that contains the allowed values of ξ. These are the spheres that are *not* obtained from the Feynman diagrams built with one propagator and two three-string vertices.

Each surface in $\mathcal{V}_{0,4}$ is a four-punctured sphere with some value of ξ. On each surface we consider the Strebel quadratic differential [6]:

$$\varphi = \phi(z)(dz)^2\,, \qquad \phi(z) = -\frac{(z^2-\xi)^2}{z^2(z-1)^2(z-\xi)^2} + \frac{a(\xi,\bar{\xi})}{z(z-1)(z-\xi)}\,. \tag{3.15}$$

Here $a(\xi,\bar{\xi})$ is a complex function of ξ *and* $\bar{\xi}$. While a is not holomorphic, we henceforth write it as $a(\xi)$, for brevity. If a is known, the quadratic differential $\phi(z)$ is fully determined. The quadratic differential has second order poles at the punctures $z = 0, 1, \xi$, and ∞. Expanding around these punctures we find

$$\begin{aligned}
\phi(z) &= -\frac{1}{z^2} + \frac{1}{z}\left(-2 - \frac{2}{\xi} + \frac{a}{\xi}\right) + \left(-3 + \frac{1}{\xi}(a-2) + \frac{1}{\xi^2}(a-3)\right) + \mathcal{O}(z)\,,\\
\phi(z) &= -\frac{1}{(z-1)^2} + \frac{1}{z-1}\left(\frac{a-2\xi}{1-\xi}\right) + \frac{a\,(\xi-2) + \xi\,(4-3\xi)}{(\xi-1)^2} + \mathcal{O}(z-1)\,,
\end{aligned}$$

$$
\begin{aligned}
\phi(z) &= -\frac{1}{(z-\xi)^2} + \frac{1}{z-\xi}\left(\frac{a-2}{\xi(\xi-1)}\right) + \frac{a-3+4\xi-2a\,\xi}{\xi^2(\xi-1)^2} + \mathcal{O}(z-\xi)\,,\\
\phi(t) &= -\frac{1}{t^2} + \frac{1}{t}\,(a-2-2\xi) + \big(a-3-2\xi+a\,\xi-3\xi^2\big) + \mathcal{O}(t)\,,
\end{aligned}
\tag{3.16}
$$

where $t = 1/z$ is used to describe the fourth puncture. Given a Strebel quadratic differential $\varphi = \phi(z)(dz)^2$ that near z_0 looks like

$$
\phi(z) = -\frac{1}{(z-z_0)^2} + \frac{r_{-1}}{(z-z_0)} + r_0 + r_1(z-z_0) + \cdots\,,
\tag{3.17}
$$

a canonical local coordinate w (defined up to a phase) is obtained by requiring $\varphi = -(1/w^2)(dw)^2$. This gives [6]:

$$
z = z_0 + \rho\, w + \frac{1}{2}r_{-1}(\rho w)^2 + \frac{1}{16}(7r_{-1}^2 + 4r_0)(\rho w)^3 + \cdots\,,
\tag{3.18}
$$

where ρ is the mapping radius, which can also be obtained using the quadratic differential. Comparing (3.8) and (3.18) we see that $\beta = r_{-1}/2$ at each punture. We can therefore use the expansions (3.16) to read:

$$
\beta_1 = \frac{a}{2\xi} - \frac{1}{\xi} - 1\,, \qquad \beta_2 = \frac{a-2\xi}{2(1-\xi)}\,, \qquad \beta_3 = \frac{a-2}{2\xi(\xi-1)}\,, \qquad \beta_4 = \frac{a}{2} - 1 - \xi\,.
\tag{3.19}
$$

Since a is a function of ξ and $\bar\xi$, all the β_I are functions of ξ and $\bar\xi$.

We can now proceed to get the values of the coordinate expansion coefficients γ in terms of a and ξ. As noted in (3.12), however, the quantity $\varepsilon = 8\beta^2 - 6\gamma$ is more useful. A short calculation shows that $\varepsilon = -(5\beta^2 + 3r_0)/2$. Reading the various values of r_0 from the expansions (3.16) and the various values of β from (3.19) we find

$$
\begin{aligned}
\varepsilon_1 &= 2 + \frac{1}{\xi}(a-2) + \frac{1}{\xi^2}\left(2 + a - \frac{5}{8}a^2\right),\\
\varepsilon_2 &= \frac{-5a^2 + 16(\xi-3) + 8a\,(\xi+3)}{8\,(\xi-1)^2}\,,\\
\varepsilon_3 &= \frac{16 + 8a - 5a^2 + 24(a-2)\xi}{8\,\xi^2\,(\xi-1)^2}\,,\\
\varepsilon_4 &= 2 + a - \frac{5}{8}\,a^2 - 2\xi + a\,\xi + 2\xi^2\,.
\end{aligned}
\tag{3.20}
$$

The function $a(\xi)$ is known numerically to high accuracy for $\xi \in \mathcal{A}$, where $\mathcal{A}$ is a specific subspace of $\mathcal{V}_{0,4}$ described in detail in [6, figures 3 and 6]. The full space $\mathcal{V}_{0,4}$ is obtained by acting on $\mathcal{A}$ with the transformations generated by $\xi \to 1-\xi$ and $\xi \to 1/\xi$, together with complex conjugation $\xi \to \bar\xi$. In fact $\mathcal{V}_{0,4}$ contains twelve copies of $\mathcal{A}$. Let $f(\mathcal{A})$ denote the region obtained by mapping each point $\xi \in \mathcal{A}$ to $f(\xi)$. Then $\mathcal{V}_{0,4}$ is composed of the six regions

$$
\mathcal{A}\,, \quad \frac{1}{\mathcal{A}}\,, \quad 1-\mathcal{A}\,, \quad \frac{1}{1-\mathcal{A}}\,, \quad 1-\frac{1}{\mathcal{A}}\,, \quad \frac{\mathcal{A}}{1-\mathcal{A}}\,,
\tag{3.21}
$$

together with their complex conjugates. The transformations $\xi \to 1-\xi$ and $\xi \to 1/\xi$ are $\mathrm{SL}(2,\mathbb{C})$ transformations that permute the points $0, 1$, and ∞. While doing so, they move the third puncture among the various regions in (3.21). The assignment of coordinates to punctures must be consistent with the $\mathrm{SL}(2,\mathbb{C})$ transformations that exchange the punctures: the quadratic differential on two conformally related surfaces must agree. For example, letting $\tilde{z} = 1 - z$, we can calculate $\phi(\tilde{z})$ using $\phi(\tilde{z})d\tilde{z}^2 = \phi(z)dz^2$. We must find that $\phi(\tilde{z})$ takes the form in (3.15) with ξ replaced by $1-\xi$, and $a(\xi)$ replaced by $a(1-\xi)$. Completely analogous remarks hold for the transformation $\tilde{z} = 1/z$. Doing these transformations explicitly we find

$$a\,(1-\xi) = 4 - a(\xi)\,, \qquad a\left(\frac{1}{\xi}\right) = \frac{a(\xi)}{\xi}\,. \tag{3.22}$$

These equations define a over the full set of regions in (3.21) once it is given on $\mathcal{A}$.

The reality of the string field theory action is guaranteed if the local coordinates on surfaces that are mirror images of each other are related by the (antiholomorphic) mirror map. Consider two four-punctured spheres: the first with uniformizer z and third puncture at ξ, the second with uniformizer $\tilde{z}$ and third puncture at $\bar{\xi}$. The antiholomorphic map relating the punctured spheres is $\tilde{z} = \bar{z}$. Two local coordinates w and $\tilde{w}$ are mirror related if $\tilde{w}(p^*) = \overline{w(p)}$, where p is a point and p^* is its image under the mirror map. In order to obtain mirror-related local coordinates the associated quadratic differentials on the surfaces must satisfy $\phi(\tilde{z}) = \overline{\phi(z)}$ [18]. It follows from (3.15) that $\phi(\tilde{z})$ takes the form indicated in this equation, with z replaced by $\tilde{z}$ and $a(\xi)$ replaced by $\overline{a(\xi)}$. We thus learn that

$$a(\,\bar{\xi}\,) = \overline{a(\xi)}\,. \tag{3.23}$$

This definition guarantees that the contribution to the amplitude of any region $\mathcal{S} \subset \mathcal{V}_{0,4}$ and the contribution from $\overline{\mathcal{S}}$ are complex conjugates of each other. Consequently, the integral over $\mathcal{V}_{0,4}$ can be done by integrating over the six regions in (3.21) and adding to the result its complex conjugate.

With $a(\xi)$ now defined over $\mathcal{V}_{0,4}$, we can also find formulae that define the mapping radii ρ_I and the coordinate expansion coefficients β_I on the various copies of $\mathcal{A}$ in terms of the values on $\mathcal{A}$. These formulae are given in appendix B.

3.3 Quartic interactions for states with $c\bar{c}$ ghost factor

In order to illustrate the earlier discussion we consider an important class of relatively simple four-point amplitudes. Suppose we want to evaluate the amplitude $\{M_1, M_2, M_3, M_4\}$ with states M_i of the form

$$|M_i\rangle = \mathcal{O}_i c_1 \bar{c}_1 |0\rangle\,, \tag{3.24}$$

where $\mathcal{O}_i$ is some expression built with matter oscillators. One can see that the ghost part of M_i is the same as that of the tachyon field. First consider the antighost insertion $\mathcal{B}\,\mathcal{B}^\star$. Since all the states have ghost oscillators with mode number one, only the $b_{-1}, \bar{b}_{-1}$ part of the antighost insertion is relevant. Using (3.14) we see that:

$$\mathcal{B}\,\mathcal{B}^\star = B^3_{-1} b^{(3)}_{-1} \qquad \overline{B^3_{-1}}\,\bar{b}^{(3)}_{-1} + \cdots\,, \tag{3.25}$$

where the dots refer to terms that are not needed in this calculation. It follows that

$$\begin{aligned}\mathcal{B}\,\mathcal{B}^\star(c_1\bar{c}_1)^{(1)}(c_1\bar{c}_1)^{(2)}(c_1\bar{c}_1)^{(3)}(c_1\bar{c}_1)^{(4)}|0\rangle &= -B^3_{-1}\overline{B^3_{-1}}(c_1\bar{c}_1)^{(1)}(c_1\bar{c}_1)^{(2)}(c_1\bar{c}_1)^{(4)}|0\rangle \\ &= -\frac{1}{\rho_3^2}(c_1\bar{c}_1)^{(1)}(c_1\bar{c}_1)^{(2)}(c_1\bar{c}_1)^{(4)}|0\rangle\,. \end{aligned} \tag{3.26}$$

Note that in our convention the states with superscripts $1, 2, 3,$ and 4 are inserted at $z = 0, 1, \xi,$ and ∞, respectively. The ghost part of the overlap is then

$$\langle\Sigma|\mathcal{B}\,\mathcal{B}^\star(c_1\bar{c}_1)^{(1)}(c_1\bar{c}_1)^{(2)}(c_1\bar{c}_1)^{(3)}(c_1\bar{c}_1)^{(4)}|0\rangle = -\frac{1}{(\rho_1\rho_2\rho_3\rho_4)^2}\langle c\bar{c}(z_1)c\bar{c}(z_2)c\bar{c}(t=0)\rangle\,, \tag{3.27}$$

where the conformal transformation of each ghost oscillator introduces a factor of the mapping radius. For the fourth puncture, which is located at $t = 1/z = 0$, the mapping radius ρ_4 refers to the t coordinate and it is a finite number. To compute the above correlator we note that

$$c\bar{c}(t=0) = \lim_{z\to\infty}\frac{1}{|z|^4}\,c\bar{c}(z)\,, \tag{3.28}$$

and therefore

$$\langle c\bar{c}(z_1)c\bar{c}(z_2)c\bar{c}(t=0)\rangle = \lim_{z\to\infty}\frac{1}{|z|^4}\langle c\bar{c}(z_1)c\bar{c}(z_2)c\bar{c}(z)\rangle = 2\lim_{z\to\infty}\frac{|z_{12}(z_1-z)(z_2-z)|^2}{|z^4|} = 2\,, \tag{3.29}$$

once we set $z_1 = 0$ and $z_2 = 1$. Back in (3.27) we find

$$\langle\Sigma|\mathcal{B}\,\mathcal{B}^\star(c_1\bar{c}_1)^{(1)}(c_1\bar{c}_1)^{(2)}(c_1\bar{c}_1)^{(3)}(c_1\bar{c}_1)^{(4)}|0\rangle = -\frac{2}{(\rho_1\rho_2\rho_3\rho_4)^2}\,. \tag{3.30}$$

The matter part of the correlator is computed using (3.8) to map the operators to the uniformizer z, at which stage the correlator is computable. On the four-punctured sphere Σ_ξ with modulus ξ we write:

$$\langle\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle\rangle_\xi \equiv \langle h_1\circ\mathcal{O}_1\ h_2\circ\mathcal{O}_2\ h_3\circ\mathcal{O}_3\ h_4\circ\mathcal{O}_4\rangle_{\Sigma_\xi}\,, \tag{3.31}$$

where the right-hand side is a matter correlator computed after the local operators $\mathcal{O}_i$ have been mapped. Our final result is therefore:

$$\{M_1, M_2, M_3, M_4\} = -\frac{2}{\pi}\int_{\mathcal{V}_{0,4}}\frac{dxdy}{(\rho_1\rho_2\rho_3\rho_4)^2}\,\langle\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle\rangle_\xi\,. \tag{3.32}$$

For the case of four zero-momentum tachyons $T = c_1\bar{c}_1|0\rangle$ the matter operators are the identity and the matter correlator is equal to one. We thus get

$$\{T^4\} = -\frac{2}{\pi}\int_{\mathcal{V}_{0,4}}\frac{dxdy}{(\rho_1\rho_2\rho_3\rho_4)^2}\,. \tag{3.33}$$

The integrand is manifestly real. Since four identical states have been inserted at the punctures, the measure to be integrated must be fully invariant under the $\mathrm{SL}(2,\mathbb{C})$ transformations that generate the six regions in (3.21). Therefore the full integral is equal to 12 times the integral over $\mathcal{A}$. This integral is easily done numerically using the data given in [6] and we recover the familiar value

$$\{T^4\} = -\frac{24}{\pi}\int_{\mathcal{A}} \frac{dxdy}{(\rho_1\rho_2\rho_3\rho_4)^2} = -72.414\,. \tag{3.34}$$

The corresponding term in the string field potential is $\kappa^2 V = \frac{1}{4!}\{T^4\}\,t^4 = -3.0172\,t^4$.

4. The explicit computation of quartic couplings

The terms a^4, a^2d^2, and d^4 in the string field potential receive contributions from cubic interactions of all levels. They also receive contributions from the elementary quartic vertex. Since a and d are marginal directions, these two types of contributions must cancel. In our earlier paper [7] we verified that this cancellation holds with good accuracy for the a^4 term. We noted that a potential complication: in closed string field theory the cubic vertex does not fully determine the quartic vertex, so the cancellation should happen for all four-string vertices that are consistent with the cubic vertex. This will happen if the integrands of the four-point amplitudes are total derivatives. In this case the quartic amplitude arises from the boundary of $\mathcal{V}_{0,4}$. This boundary is completely determined by the geometry of the three-string vertex: gauge invariance requires that the boundary of $\mathcal{V}_{0,4}$ match the configurations obtained with two cubic vertices joined by a collapsed propagator. Letting G and D denote the states associated with a and d respectively (see (2.4)), the integrands in $\{G^4\}$, $\{G^2D^2\}$, and $\{D^4\}$ are thus expected to be total derivatives. In our earlier paper we confirmed that the integrand in $\{G^4\}$ is a total derivative. We will do the same here for the other two amplitudes.

The computations to be discussed below determine the quartic contribution to the potential. The results we obtained are summarized by

$$\kappa^2 V_4(a,d) = -0.2560\,a^4 + 0.4571\,a^2d^2 - 0.1056\,d^4\,. \tag{4.1}$$

4.1 Elementary contribution to a^2d^2

In order to compute this amplitude we insert G at the first and fourth punctures ($z=0$ and $z=\infty$, respectively) and D at the second and third punctures ($z=1$ and $z=\xi$, respectively). We begin our analysis with the computation of the ghost part of the amplitude.

Consider the antighost insertion $\mathcal{B}\mathcal{B}^\star$ acting on the ghost part of the four states:

$$\mathcal{B}\,\mathcal{B}^\star(c_1\bar{c}_1)^{(1)}(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})^{(2)}(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})^{(3)}(c_1\bar{c}_1)^{(4)}|0\rangle\,. \tag{4.2}$$

Since punctures one and four are fixed and the states inserted in them have ghost oscillators $c_1\bar{c}_1$, the antighost factor $\mathcal{B}\mathcal{B}^\star$ is only supported on punctures two and three. A small calculation shows that

$$\mathcal{B}\,\mathcal{B}^\star(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})^{(2)}(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})^{(3)} = (C_1^2\overline{C_1^3} - B_1^2\overline{B_1^3})c_1^{(2)}\bar{c}_1^{(3)} + B_1^2\overline{B_{-1}^3}\,c_1^{(2)}\bar{c}_{-1}^{(3)} + \star\text{-conj}\,. \tag{4.3}$$

Both sides of this equation have vacuum states to the right, which have not been written to avoid clutter. It follows that

$$\begin{aligned}\langle\Sigma|\mathcal{B}\mathcal{B}^\star|T\rangle|D\rangle|D\rangle|T\rangle = (C_1^2\overline{C_1^3} - B_1^2\overline{B_1^3})\,\Big\langle (c_1\bar{c}_1)^{(1)}\,, c_1^{(2)}\,, \bar{c}_1^{(3)}\,, (c_1\bar{c}_1)^{(4)}\Big\rangle + \\ + B_1^2\overline{B_{-1}^3}\,\Big\langle (c_1\bar{c}_1)^{(1)}\,, c_1^{(2)}\,, \bar{c}_{-1}^{(3)}\,, (c_1\bar{c}_1)^{(4)}\Big\rangle + *\text{-conj}\,. \qquad (4.4)\end{aligned}$$

Note that the star-conjugate insertions give rise to complex conjugate correlators. This happens because all other ghost states in the correlator are self-conjugate. Two correlators are thus needed to evaluate the (4.4). The first arises from the first line on the right-hand side

$$\Big\langle (c_1\bar{c}_1)^{(1)}\,, c_1^{(2)}\,, \bar{c}_1^{(3)}\,, (c_1\bar{c}_1)^{(4)}\Big\rangle = \frac{2\bar{\xi}}{\rho_1^2\rho_2\rho_3\rho_4^2}\,. \qquad (4.5)$$

The second correlator appears on the second line of (4.4). It involves the state created by $\bar{c}_{-1}^{(3)}$ on the vacuum. The corresponding operator $\frac{1}{2}\bar{\partial}^2\bar{c}$ has a nontrivial transformation under a conformal map:

$$\frac{1}{2}\bar{\partial}^2\bar{c}(\bar{w}) = \rho_I\left(\frac{1}{2}\bar{\partial}^2\bar{c}(\bar{z}_I) - \bar{\beta}_I\bar{\partial}\bar{c}(\bar{z}_I) + \frac{\bar{\varepsilon}_I}{2}\bar{c}(\bar{z}_I)\right). \qquad (4.6)$$

With the help of this transformation we find

$$\Big\langle (c_1\bar{c}_1)^{(1)}\,, c_1^{(2)}\,, \bar{c}_{-1}^{(3)}\,, (c_1\bar{c}_1)^{(4)}\Big\rangle = \frac{\rho_3}{\rho_1^2\rho_2\rho_4^2}\,(\bar{\varepsilon}_3\bar{\xi} - 2\bar{\beta}_3)\,. \qquad (4.7)$$

Finally, we simplify the expressions in (4.4) that depend on the coefficients B and C:

$$C_1^2\overline{C_1^3} - B_1^2\overline{B_1^3} = \rho_2\rho_3\Big(\bar{\partial}\beta_2\partial\bar{\beta}_3 - \partial\beta_2\bar{\partial}\bar{\beta}_3 - \frac{1}{2}\bar{\varepsilon}_3\partial\beta_2\Big)\,, \qquad B_1^2\overline{B_{-1}^3} = \frac{\rho_2}{\rho_3}\,\partial\beta_2\,. \qquad (4.8)$$

Using the above results (4.4) simplifies down to

$$\langle\Sigma|\mathcal{B}\mathcal{B}^\star|T\rangle|D\rangle|D\rangle|T\rangle = \frac{2}{(\rho_1\rho_4)^2}\,\big(\bar{\partial}\beta_2\partial(\bar{\xi}\bar{\beta}_3) - \partial\beta_2\bar{\partial}(\bar{\xi}\bar{\beta}_3) + *\text{-conj}\big)\,. \qquad (4.9)$$

This concludes our computation of the ghost part of the integrand.

The matter part of the integrand is much simpler. We have two G's one at $z = 0$ and one at $z = \infty$. A short computation gives

$$\langle\langle(\partial X\bar{\partial}X)^{(1)}(\partial X\bar{\partial}X)^{(4)}\rangle\rangle = (\rho_1\rho_4)^2\,. \qquad (4.10)$$

Note that the powers of mapping radii cancel out in the product of ghost and matter amplitudes. Making use of (3.6) the four-point amplitude is

$$\{G^2D^2\} = \frac{4}{\pi}\int_{\mathcal{V}_{0,4}} dxdy\;\,\mathrm{Re}\,\big(\bar{\partial}\beta_2\partial(\bar{\xi}\bar{\beta}_3) - \partial\beta_2\bar{\partial}(\bar{\xi}\bar{\beta}_3)\big)\,. \qquad (4.11)$$

Since we have the same states on punctures one and four, and these punctures are exchanged by the transformation $z \to 1/z$, the integral over $\mathcal{A}$ gives the same contribution as the integral over $1/\mathcal{A}$. This follows from the $\mathrm{SL}(2,\mathbb{C})$ invariance of the construction, but can also be checked explicitly using the formulae given in Appendix B. Since the integrand is already real, conjugate regions give identical contributions. Consequently, the four regions $\mathcal{A}$, $1/\mathcal{A}$, $\overline{\mathcal{A}}$, and $1/\overline{\mathcal{A}}$ all give the same contribution. To get the full amplitude we must multiply the contributions of $\mathcal{A}$, of $1-\mathcal{A}$ and $1-1/\mathcal{A}$ by four: Therefore, the full amplitude is:

$$\begin{aligned}\{G^2D^2\} &= 4\cdot\frac{4}{\pi}\left[\int_{\mathcal{A}}+\int_{1-\mathcal{A}}+\int_{1-1/\mathcal{A}}\right] dxdy\ \mathrm{Re}\left(\bar\partial\beta_2\partial(\bar\xi\bar\beta_3)-\partial\beta_2\bar\partial(\bar\xi\bar\beta_3)\right)\\ &\simeq 4\,(-0.03122+0.09671+0.39157)=1.8283\,. \end{aligned} \tag{4.12}$$

In the above, the second and third integrals were evaluated by pulling back the integrands into $\mathcal{A}$, where all relevant functions are known numerically. The details are given in Appendix B. The contribution to the potential is

$$\kappa^2 V=\frac{6}{4!}\{G^2D^2\}\,a^2d^2=0.4571\,a^2d^2\,. \tag{4.13}$$

We now verify that the integrand of the $\{G^2D^2\}$ amplitude is a total derivative. Indeed, a short computation shows that

$$\left(\bar\partial\beta_2\partial(\bar\xi\bar\beta_3)-\partial\beta_2\bar\partial(\bar\xi\bar\beta_3)\right)\,d\xi\wedge d\bar\xi=d\,\Omega^{(1)}\,, \tag{4.14}$$

where

$$\Omega^{(1)}=\frac{1}{2}\left[-\beta_2\partial(\bar\xi\bar\beta_3)+(\partial\beta_2)\,\bar\xi\bar\beta_3\right]\,d\xi+\frac{1}{2}\left[(\bar\partial\beta_2)\,\bar\xi\bar\beta_3-\beta_2\bar\partial(\bar\xi\bar\beta_3)\right]\,d\bar\xi\,. \tag{4.15}$$

Since $a(\xi)$, $\beta_2(\xi)$, and $\beta_3(\xi)$ are all regular functions on $\mathcal{V}_{0,4}$, we see that $\Omega^{(1)}$ is a well-defined one-form. The amplitude reduces to the integral of $\Omega^{(1)}$ over the boundary of $\mathcal{V}_{0,4}$.

4.2 Elementary contribution to d^4

For the amplitude $\{D^4\}$ the matter correlator is just one but the ghost correlator is quite nontrivial. The antighost insertion $\mathcal{B}\mathcal{B}^\star$ acts on four dilaton states. Ghost number conservation implies that the only nonvanishing correlators are those in which the antighost insertion supplies one b oscillator and one $\bar b$ oscillator. Note also that the term $b^{(3)}_{-1}\bar b^{(3)}_{-1}$ does not contribute. Making use of (3.14) the relevant terms in the antighost insertion are

$$\mathcal{B}\mathcal{B}^\star=\left(B^3_{-1}b^{(3)}_{-1}\sum_{I\neq 3}\overline{B^I_1}\,\bar b^{(I)}_1+\star\text{-conj.}\right)+\sum_{I\neq J}M^{IJ}\,b^{(I)}_1\bar b^{(J)}_1\,,\qquad M^{IJ}\equiv B^I_1\overline{B^J_1}-C^I_1\overline{C^J_1}\,. \tag{4.16}$$

Acting on the four dilatons and forming the correlator,

$$\langle\Sigma|\mathcal{B}\mathcal{B}^\star|D^4\rangle=\left(\sum_{I\neq 3}B^3_{-1}\overline{B^I_1}\,\left\langle D,D,c^{(3)}_{-1},\bar c^{(I)}_1\right\rangle+\text{c.c.}\right)-\sum_{I\neq J}M^{IJ}\langle D,D,c^{(I)}_1,\bar c^{(J)}_1\rangle$$

$$= - \sum_{I\neq J\neq K\neq 3} B^3_{-1}\overline{B^I_1}\ \left\langle (c_{-1}c_1)^{(J)}, (\bar{c}_{-1}\bar{c}_1)^{(K)}, c^{(3)}_{-1},\, \bar{c}^{(I)}_1 \right\rangle + \text{c.c.}$$
$$+ \sum_{I\neq J\neq K\neq L} M^{IJ}\ \left\langle (c_{-1}c_1)^{(K)}, (\bar{c}_{-1}\bar{c}_1)^{(L)}, c^{(I)}_1, \bar{c}^{(J)}_1 \right\rangle . \quad (4.17)$$

In order to complete our calculation we must evaluate the two correlators that appear on the final right-hand side. Defining

$$A_{IJ} \equiv \left\langle \, (c_{-1}c_1)^{(I)} \, , c^{(J)}_{-1} \right\rangle , \qquad B_{IJ} \equiv \left\langle \, (c_{-1}c_1)^{(I)} \, , c^{(J)}_1 \right\rangle , \quad (4.18)$$

the amplitude in (4.17) becomes

$$\langle \Sigma|\mathcal{B}\mathcal{B}^\star|D^4\rangle = \sum_{I\neq J\neq K\neq 3} 2B^3_{-1}\left(\overline{B^I_1}A_{J3}\overline{B_{KI}} + \text{c.c.}\right) - \sum_{I\neq J\neq K\neq L} 2M^{IJ}\ B_{KI}\overline{B_{LJ}} . \quad (4.19)$$

It just remains to evaluate A_{IJ} and B_{IJ}. There is one small complication here. Since we treat the fourth puncture asymmetrically we must distinguish the case when I or J are equal to four. We find that for $I, J \neq 4$

$$\begin{aligned} A_{IJ} &= \rho_J\left(\beta_J - \beta_I - 2\beta_I\beta_J z_{IJ} + \frac{1}{2}\varepsilon_J z_{IJ}(1-\beta_I z_{IJ})\right) \\ B_{IJ} &= \frac{1}{\rho_J}\, z_{IJ}\,(1-\beta_I\, z_{IJ})\,, \qquad \text{for} \quad I, J\neq 4 \, . \end{aligned} \quad (4.20)$$

Here $z_{IJ} = z_I - z_J$. We also need the following special values which arise when one of the states is located at the fourth puncture ($z = \infty$):

$$A_{4J} = \rho_J\left(\,\frac{1}{2}\,\varepsilon_J(\beta_4+z_J) - \beta_J\right), \qquad B_{I4} = \frac{\beta_I}{\rho_4}\,, \qquad B_{4J} = \frac{1}{\rho_J}(z_J+\beta_4)\,. \quad (4.21)$$

In these equations I and J are different from four. It is possible to obtain (4.21) by taking a suitable limit of (4.20). One must let

$$\beta_4 \to \frac{1}{z_4} - \frac{\beta_4}{z_4^2}\,, \qquad \text{and} \qquad \rho_4 \to z_4^2\,\rho_4\,, \quad (4.22)$$

and then take the limit as $z_4 \to \infty$. The above replacements are the ones involved in changing uniformizer from z to $t = 1/z$.

Having calculated explicitly all the quantities that enter into (4.19), we can do the numerical integration:

$$\{D^4\} = \frac{12}{\pi}\int_{\mathcal{A}} dxdy\ \langle \Sigma|\mathcal{B}\,\mathcal{B}^\star|D^4\rangle = -2.5336\,. \quad (4.23)$$

The contribution to the potential is therefore

$$\kappa^2 V = \frac{1}{4!}\{D^4\}\, d^4 = -0.1056\, d^4\,. \quad (4.24)$$

In order to demonstrate that the integrand is a total derivative, we have shown that $\langle \Sigma|\mathcal{B}\mathcal{B}^\star|D^4\rangle = \partial f + \bar\partial \bar f$ for a suitable function f of ξ and $\bar\xi$. As a result, the two-form integrand (up to overall constants) is indeed exact: $d\xi \wedge d\bar\xi\, (\partial f + \bar\partial \bar f) = d\,(f d\xi - \bar f d\bar\xi)$. The calculation of f is quite laborious and is best done using equation (4.20) for all values of I and J. The result is then:

$$\begin{aligned} f(\xi,\bar\xi) = -\sum_{I\neq J\neq K\neq L} \Big\{ & \beta_J \bar\partial\bar\beta_I z_{JK}\bar z_{IL} - \beta_J\bar\beta_I z_{JK}\delta_{3L} + \beta_J\bar\beta_I z_{JK}\delta_{3I} + \\ & + \frac{1}{2}\beta_J\beta_K\bar\partial\bar\beta_I z_{JK}^2 \bar z_{IL} + \frac{1}{2}\beta_J\beta_K\bar\beta_I z_{JK}^2\delta_{3I} - \frac{1}{2}\beta_J\beta_K\bar\beta_I z_{JK}^2\delta_{3L} + \\ & + \beta_J\bar\beta_L\bar\partial\bar\beta_I z_{JK}\bar z_{IL}^2 + 2\beta_J\bar\beta_I\bar\beta_L z_{JK}\bar z_{IL}\delta_{3I} + \frac{1}{2}\beta_J\beta_K\bar\beta_L\bar\partial\bar\beta_I z_{JK}^2\bar z_{IL}^2 + \\ & + \beta_J\beta_K\bar\beta_I\bar\beta_L z_{JK}^2\bar z_{IL}\delta_{3I} - \bar\beta_J\bar\partial\beta_I z_{IL}\bar z_{JK} - \frac{1}{2}\bar\beta_J\bar\beta_K\bar\partial\beta_I z_{IL}\bar z_{JK}^2 - \\ & - \beta_L\bar\beta_J\bar\partial\beta_I z_{IL}^2\bar z_{JK} - \frac{1}{2}\beta_L\bar\beta_J\bar\beta_K\bar\partial\beta_I z_{IL}^2\bar z_{JK}^2 \Big\}. \end{aligned} \tag{4.25}$$

In this expression all β_4 must be replaced as indicated in (4.22). The final answer is finite and well defined over $\mathcal{V}_{0,4}$.

5. Analysis and conclusions

In this final section we discuss our results. We first examine the cancellations between cubic and quartic terms, using certain projections of the cubic data in order to estimate the effects of terms that have not been computed. We then attempt to give a definition of level suitable for quartic interactions. While we are not able to give convincing evidence for any specific definition, we find out that the level suppression observed for cubic interactions seems to extend to quartic interactions. This is good news, as it suggests that computations carried out with cubic and quartic interactions would converge as the level is increased.

5.1 Cancellations and fits

In this work we checked explicitly the quartic structure of closed string field theory. The existence of flat directions implies that the infinite-level cubic contribution to the effective potential for a and d must be cancelled by elementary quartic interactions. We claim that the potential $\mathbb{V}(a,d)$, defined as

$$\mathbb{V}(a,d) \equiv \lim_{\ell\to\infty} \mathbb{V}_{(\ell)}(a,d) + V_4(a,d)\,, \tag{5.1}$$

should vanish identically. Let us see how well we have checked this. The relevant data has been collected on table 3. Even at $\ell = 6$ the pattern of cancellations is quite clear. For a^4 the quartic term cancels 86% of the cubic answer. For a^2d^2 and d^4 the quartic interactions cancel about 90% of the cubic answers.

Consider now the same cancellations using projections from the data of cubic computations. As we explained in ref. [7], a fit for the coefficient $\mathcal{C}_{a^4}$ using the best available data gives:

$$\mathcal{C}_{a^4}(\ell) \simeq 0.25585 + \frac{0.50581}{\ell^2} + \frac{1.06366}{\ell^3}\,, \tag{5.2}$$

level/vertex	$\mathcal{C}_{a^4}$	$\mathcal{C}_{a^2d^2}$	$\mathcal{C}_{d^4}$
$\ell = 6$, cubic	0.2978	-0.5065	0.1163
$\ell = \infty$, projected	0.2559	-0.4488	0.1044
elementary quartic	-0.2560	0.4571	-0.1056

Table 3: The coefficients of the quartic terms in the effective potential for a and d. First row: result from cubic interactions integrating massive fields up to level six. Second row: projected result from cubic interactions to all levels. Third row: elementary quartic contributions, read from (4.1).

The above fit was implied by the fit with leading $1/\ell$ of the related open string coefficients. As remarked in [7], the projected value 0.25585 that follows from (5.2) and the elementary quartic value (-0.25598) essentially agree perfectly.

We do not have a priori arguments that tell what kind of fit should be used for $\mathcal{C}_{d^4}$. Equation (5.2) would suggest a $1/\ell^2$ fit, and this gives a somewhat low projection (0.09374). We thus attempted a fit to $1/\ell^3$, which works very well using the data for $\ell = 4$ and $\ell = 6$ in table 1:

$$\mathcal{C}_{d^4}(\ell) \simeq 0.1044 + \frac{2.5647}{\ell^3}\,. \tag{5.3}$$

The projection 0.1044 is cancelled by the elementary quartic term (-0.1056) to an accuracy of 1.1%. As a check of the plausibility of (5.3) we attempted a fit of the form $f_0 + f_1/\ell^\gamma$ and adjusted γ so that f_0 matches precisely the elementary quartic term. This gives $\gamma \sim 3.2$, which is reasonably close to our guess $\gamma = 3$. As a further check we use the data of table 2. Since the contribution for $\ell = 8$ is exceptional, we only use the data for $\ell = 4, 6$, and 10. This time we get

$$\mathcal{C}_{d^4}(\ell) \simeq 0.1058 + \frac{2.4593}{\ell^3}\,. \tag{5.4}$$

This projection is exceptionally good, it cancels the elementary quartic term to an accuracy of 0.2%.

We have no guidance for $\mathcal{C}_{a^2d^2}$, either. It seems reasonable to take a level dependence somewhere in between those of $\mathcal{C}_{a^4}$ and $\mathcal{C}_{d^4}$. We thus considered a fit with $\ell^{-5/2}$ finding:

$$\mathcal{C}_{a^2d^2}(\ell) \simeq -0.4488 - \frac{5.0880}{\ell^{5/2}}\,. \tag{5.5}$$

The projection (-0.4488) is cancelled by the elementary quartic term (0.4571) to an accuracy of 1.8%. We also found that the match is perfect for a fit with $\ell^{-\gamma}$ with $\gamma \simeq 2.7$. All in all, we believe that the above results are good evidence that the elementary quartic amplitudes of marginal operators have been computed correctly.

The vanishing of the coefficient of a^2d^2 in the effective potential confirms the expectation that the marginal directions a and d in fact generate a two dimensional space of marginal directions. An effective potential with vanishing a^4 and d^4 terms but non-vanishing a^2d^2 would be consistent with the existence of marginal directions — but no two-dimensional moduli space. It is thus interesting to visualize the increasing flatness of the effective potential on the two dimensional space. Since the calculated potentials are, to this order, invariant under the separate tranformations $a \to -a$ and $d \to -d$, it suffices

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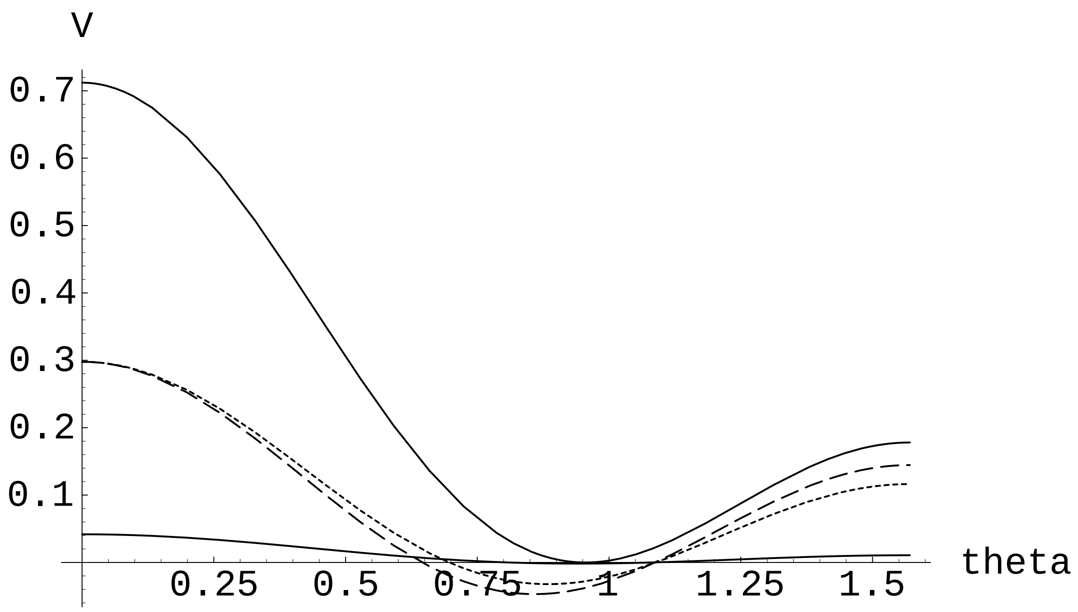

Figure 3: The potential $\mathbb{V}(a = \cos\theta, d = \sin\theta)$ plotted as a function of θ. We show the potentials $\mathbb{V}_{(0)}$ (highest curve for $\theta = 0$), $\mathbb{V}_{(4)}$ (the dashed line), $\mathbb{V}_{(6)}$ (the dotted line), and the potential including the quartic elementary interactions.

to consider the potential on the first quadrant of the (a, d) plane. Since the potential scales uniformly when $(a, d) \to (\lambda a, \lambda d)$, it suffices to examine the potential along the arc of a unit circle on the first quadrant. Letting $a = \cos\theta$ and $d = \sin\theta$, we can examine the potentials $\mathbb{V}(a, d)$ as $\theta \in [0, \pi/2]$. For this we have produced figure 3, which plots the potentials $\mathbb{V}_{(\ell)}(\cos\theta, \sin\theta)$ as a function of θ. The solid top curve is $\mathbb{V}_{(0)}(\cos\theta, \sin\theta)$, which happens to be a perfect square and vanishes for $\theta = \tan^{-1}(\sqrt{2}) \simeq 0.955$. While the coefficients $\mathcal{C}_{\ldots}(\ell)$ decrease as we increase ℓ, the potentials do not approach zero uniformly as a function of θ: we do *not* have $|\mathbb{V}_{(0)}(\theta)| \geq |\mathbb{V}_{(4)}(\theta)| \geq |\mathbb{V}_{(6)}(\theta)|$. This is especially clear for a range of θ values near to but smaller than one. The final curve we show is $\mathbb{V}_{(6)} + V_4$. This curve, solid and near the horizontal axis, makes it clear that even without projection of the cubic data the effective potential has become rather small.

The elementary quartic contribution to a^4 (of value -0.2560) is roughly of the magnitude that would be produced by integrating out fields of level 6. This is suggested by table 1, where we see that the contribution from fields of level 4 are larger than the quartic contribution, and contributions from level 8 (of value -0.0319) are considerably smaller (in this rough argument it is not relevant that the contribution of level six fields is actually zero). On the other hand, both for $a^2 d^2$ and d^4, the quartic contributions are larger than those that arise from integrating level four fields, though significantly smaller than those from integrating out the tachyon. We can safely state that relevant *quartic interactions must be included as the level of the string field is two or higher.* For the case at hand, it means including the quartic interactions after integrating out the tachyon in the cubic action.

5.2 Quartic suppression and attempts to define level

Here we discuss possible definitions of level for quartic interactions. A fully successful definition would have the following properties:

1. It should furnish a meaningful comparison between quartic interactions: contributions to calculable quantities from quartic interactions should be suppressed as the level of the interactions increases.

2. It should furnish a meaningful comparison between cubic and quartic interactions: cubic and quartic interactions of the same level should have roughly equal contributions to calculable quantities.

Needless to say, it is easier to satisfy 1 than it is to satisfy 2. Since it is not understood, even in open string field theory, why level expansion works, we will focus on a property that is expected to play some role: amplitudes have level-dependent powers of mapping radii that give rise to exponential suppression. This is clearly not the full story in level expansion, where convergence is typically characterized by corrections with inverse powers of level [19, 20]. We thus consider the following a first attempt on a difficult problem.

We begin first with cubic interactions. We also consider, for simplicity, three identical states of level ℓ. We will assume that the leading level dependence of the three-string coupling is due to the dimension of the state and the associated conformal map; this gives a factor $\mathcal{R}^{-(L_0+\bar{L}_0)} = \mathcal{R}^{-\ell+2}$ for each state. This is the leading level dependence if, (1) the 3-point correlator has at most power dependence on ℓ and, (2) the contributions that arise because the operator is not primary are also suppressed. It is natural to assume that the level of a cubic interaction in closed string field theory is defined just like in open string field theory, by the sum of levels of the fields that are coupled. Then, in our present case, the cubic term in the action has level $L = 3\ell$ and is of the form $\frac{1}{3!}c_3\phi^3$, with

$$c_3(L) \sim \mathcal{R}^{-3\ell+6} = \mathcal{R}^{-L+6} = \exp\left(-L\ln\mathcal{R} + 6\ln\mathcal{R}\right), \qquad \ln\mathcal{R} \simeq 0.2616\,. \tag{5.6}$$

The exponential suppression due to level is striking. Even the growth in the number of states cannot match it: the number of states grows like $\exp(a_0\sqrt{L})$, where a_0 is a finite positive constant. In terms of ℓ, the above reads

$$c_3 \sim \exp\left(-0.7849\,\ell + 1.5697\right). \tag{5.7}$$

The ℓ-independent constant in the exponent should *not* be trusted since our assumptions ignored all powers of ℓ and all constants. If included, c_3 takes the correct value for three (level zero) tachyons. So really, we have

$$c_3 \sim \exp(-0.2616L)\,. \tag{5.8}$$

Let's now consider the elementary four-point interaction of four identical level l states. These would go like $\frac{1}{4!}c_4\phi^4$ where, under assumptions similar to those stated before, the value of c_4 is roughly

$$c_4 \simeq \frac{24}{\pi}\int_{\mathcal{A}} dxdy\,\left(\rho_1\rho_2\rho_3\rho_4\right)^{\ell-2}. \tag{5.9}$$

The above formula is the obvious generalization of the tachyon quartic amplitude (3.34), with which it agrees when $\ell = 0$. We have computed numerically c_4 for various values of $\ell \geq 0$ and, interestingly, the results are well fit by a decaying exponential:

$$c_4 \simeq \exp\left(-1.135\,\ell + 4.27\,\right). \tag{5.10}$$

Again, the constant term in the exponential is not reliable and is only included in order to give the correct answer for the coupling of tachyons. We have thus learned that

$$c_4 \simeq \exp\left(-1.135\,\ell\right). \tag{5.11}$$

The level L of a four-string interaction increases with ℓ so this result suggests that quartic interactions are suppressed as the level is increased, the statement of the first condition given at the beginning of this subsection. This is grounds for optimism.

How should we define the level L of a four string elementary interaction ? One natural option would be to take

$$L = \alpha + \beta \sum_{i=1}^{4} \ell_i\,, \tag{5.12}$$

with α and β constants to be determined. If we take $\alpha = 0$ and $\beta = 1$, the simplest generalization of level for cubic interactions, we get $L = 4\ell$ and (5.11) becomes $c_4 \sim \exp(-0.2838L)$, which is intriguingly similar to (5.8). Thus, for $L = 4\ell$ we get $c_3(L) \sim c_4(L)$.

It is not clear, however, that similar levels should lead to $c_3 \sim c_4$. Effective potentials, for example, suggest that c_3^2 and c_4 give similar contributions to observables. So, in the spirit of condition 2 we can require that for similar levels we get similar contributions:

$$(c_3(L))^2 \sim c_4(L)\,. \tag{5.13}$$

Focusing on level dependent terms and using (5.8), (5.11), and $L \sim 4\beta\ell$ we find $\beta \simeq 0.54$. This suggests $L \sim \frac{1}{2}\sum_i \ell_i$ for quartic interactions. We do not have sufficient data to test the validity of such relation. One may attempt to find the value of α in (5.12) but that requires a control over level independent terms in our expansions that we do not have. Using (5.7) and (5.10) at face value would give $\alpha \simeq -2.2$. Ideally we would wish $\alpha \sim 2$, which would add level to quartic interactions. The negative α we find is just a reflection of the fact that the quartic tachyon amplitude is surprisingly big. In fact, it is so big that it eliminates the critical point in the potential calculated with quadratic and cubic terms [3]. Our computations with marginal directions have suggested a much more benign behavior, one in which quartic contributions are suppressed with respect to the leading contributions from the cubic term.

We think that the outlook for level expansion in closed string field theory is positive. The above estimates suggest that the same reasons that make higher level cubic interactions suppressed also make higher level quartic interactions suppressed. The following strategy would then seem safe: calculate with cubic interactions to high level until convergence is clear and a result A_3 is obtained. Then add quartic interactions increasing the level until convergence occurs again, this time obtaining a corrected result A_4. Continue in this way with quintic and higher order interactions to obtain quantities $A_5, A_6, \ldots$. Throughout the process we want $A_{i+1} \sim A_i$. The final result is the limit of A_n as $n \to \infty$. In carrying out these calculations one would hope that at each time one begins a computation including terms of one order higher, there is a set of low-level interactions at that new order such that the result obtained including them does not differ greatly from the result obtained without them.

We have seen that in the present calculations it makes sense to add quartic interactions once the string field includes states of level two or higher. This strategy helps produce clearer convergence. We do not know, in general, at what point quartic interactions should be added. Since our present analysis suggests that convergence will occur anyway, such determination may not be crucial. When the computation of cubic interactions is inexpensive, we may compute a large number of them before including quartic terms.

5.3 Open questions

There are several questions that we have not addressed. We have not attempted to discuss *large* marginal deformations nor *large* dilaton deformations. In level-expanded open string field theory the Wilson line deformation parameter encounters an obstruction for a finite value [13]. In closed string field theory we have seen (section 2.1) that after integrating out the cubic tachyon interactions the radius deformation parameter a has a finite range, while the dilaton deformation has an infinite range. Finite ranges appear when the solution (marginal) branch meets another branch of the equations of motion. Since higher level and higher order interactions imply equations of motion of higher order, the results obtained with the lowest level cubic interactions may be modified. Or perhaps not. We have computed the effects of integrating out the cubic couplings for the lowest level *massive* closed string fields. For ranges in which the computations are reliable we found no evidence of a limiting value for the dilaton. This and further results will be published in [21].

If closed string field theory were able to define dilaton deformations that correspond to infinite string coupling it would be a very exciting result. For superstrings it would imply that a (yet to be constructed) type IIA quantum closed superstring field theory would contain M-theory in its configuration space. Most likely infinite string coupling would correspond to $d = \infty$. If infinite coupling could be reached for some finite value $d = d_0$ the situation might be even better: M-theory would be obtained as a regular expansion around a point clearly inside the configuration space of the string field theory.[1]

Our increased computational ability, due largely to the results of Moeller, and the experience gained in this paper make it interesting to reconsider the bulk closed string tachyon potential [21]. We are now able to compute fairly efficiently any set of quartic interactions. While the effective descriptions based on conformal field theory indicate that the dilaton potential is just a multiplicative factor for the bulk tachyon potential, this is not the case in string field theory. The string field dilaton and the sigma model dilaton

[1]We thank W. Taylor for suggesting this possibility.

are related by field redefinitions that involve the tachyon and other massive fields. An investigation of the bulk tachyon potential in string field theory requires the inclusion of the dilaton and the computation of its off-shell couplings to other massive fields. Only now we have the technology to do this.

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A. Cubic potentials

The potential for the level-four string field in (2.12) is

$$\begin{aligned}\kappa^2 V_{(4)} &= (f_1)^2 + f_1\left(\frac{121}{432}a^2 - \frac{1}{96}d^2\right) + \frac{625}{4}(f_2)^2 + f_2\left(\frac{15625}{1728}a^2 - \frac{15625}{3456}d^2\right) - \\ &\quad - \frac{25}{2}\left[(f_3)^2 + (f_4)^2\right] - (f_3+f_4)\left(\frac{1375}{864}a^2 - \frac{125}{576}d^2\right) + \\ &\quad + 4(r_1)^2 + r_1\left(\frac{27}{16}a^2 - \frac{25}{864}d^2\right) - 2\left[(r_2)^2 + (r_3)^2\right] + (r_2+r_3)\left(\frac{11}{16}a^2 + \frac{5}{288}d^2\right) + \\ &\quad + 25\left[(r_4)^2 + (r_5)^2\right] - (r_4+r_5)\left(\frac{125}{32}a^2 + \frac{625}{1728}d^2\right) + 2r_6r_7 - \frac{8}{27}(r_7 - r_6)ad\,. \end{aligned} \tag{A.1}$$

The level six string field needed for the computation of quartic terms in the potential for a and d is

$$\begin{aligned}|\Psi_6\rangle &= h_1 c_{-2}\bar{c}_{-2} + h_2\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-2}\bar{\alpha}_{-1}c_1\bar{c}_1 + h_3 b_{-2}c_{-1}c_1\bar{b}_{-2}\bar{c}_{-1}\bar{c}_1 + \\ &\quad + (g_1\alpha_{-2}\alpha_{-1}c_1\bar{c}_{-2} + \tilde{g}_1\bar{\alpha}_{-2}\bar{\alpha}_{-1}c_{-2}\bar{c}_1) + (g_2 b_{-2}c_{-1}c_1\bar{c}_{-2} + \tilde{g}_2 c_{-2}\bar{b}_{-2}\bar{c}_{-1}\bar{c}_1) + \\ &\quad + g_3\alpha_{-2}\alpha_{-1}c_1\bar{b}_{-2}\bar{c}_{-1}\bar{c}_1 + \tilde{g}_3\bar{\alpha}_{-2}\bar{\alpha}_{-1}b_{-2}c_{-1}c_1\bar{c}_1 + \\ &\quad + (g_4\alpha_{-2}\bar{\alpha}_{-2}c_{-1}c_1 + \tilde{g}_4\alpha_{-2}\bar{\alpha}_{-2}\bar{c}_{-1}\bar{c}_1) + (g_5\alpha_{-1}\bar{\alpha}_{-2}c_{-2}c_1 + \tilde{g}_5\alpha_{-2}\bar{\alpha}_{-1}\bar{c}_{-2}\bar{c}_1) + \\ &\quad + g_6\alpha_{-2}\bar{\alpha}_{-1}c_{-1}c_1\bar{b}_{-2}\bar{c}_1 + \tilde{g}_6\alpha_{-1}\bar{\alpha}_{-2}b_{-2}c_1\bar{c}_{-1}\bar{c}_1 + \\ &\quad + g_7\alpha_{-1}\bar{\alpha}_{-1}c_{-2}c_1\bar{b}_{-2}\bar{c}_1 + \tilde{g}_7\alpha_{-1}\bar{\alpha}_{-1}b_{-2}c_1\bar{c}_{-2}\bar{c}_1 + \\ &\quad + g_8 b_{-2}\bar{c}_{-2}\bar{c}_{-1}\bar{c}_1 + \tilde{g}_8 c_{-2}c_{-1}c_1\bar{b}_{-2}\,. \end{aligned} \tag{A.2}$$

This string field contains states of ghost numbers $(-1,3)$ and $(3,-1)$. The associated potential is

$$\begin{aligned}\kappa^2 V_{(6)} &= 4h_1h_3 + 4g_2\tilde{g}_2 + 16g_4\tilde{g}_4 + 8g_5\tilde{g}_6 + 8g_6\tilde{g}_5 + 4g_7\tilde{g}_7 + 4g_8\tilde{g}_8 + \\ &\quad + \frac{50}{729}h_1d^2 + \frac{200}{729}h_3d^2 + \frac{100}{729}(g_2+\tilde{g}_2)d^2 - \frac{200}{729}(g_8+\tilde{g}_8)d^2 + \\ &\quad + \frac{128}{729}(g_4-\tilde{g}_4)ad - \frac{160}{729}(g_5-\tilde{g}_5)ad - \frac{320}{729}(g_6-\tilde{g}_6)ad + \frac{400}{729}(g_7-\tilde{g}_7)ad\,. \end{aligned} \tag{A.3}$$

B. Transformation laws

We record the following transformation laws:

$$
\begin{aligned}
\beta_1(1-\xi) &= -\beta_2(\xi)\,, & \rho_1(1-\xi) &= \rho_2(\xi)\,,\\
\beta_2(1-\xi) &= -\beta_1(\xi)\,, & \rho_2(1-\xi) &= \rho_1(\xi)\,,\\
\beta_3(1-\xi) &= -\beta_3(\xi)\,, & \rho_3(1-\xi) &= \rho_3(\xi)\,,\\
\beta_4(1-\xi) &= -\beta_4(\xi)-1\,, & \rho_4(1-\xi) &= \rho_4(\xi)\,,\\
\beta_1\left(\frac{1}{\xi}\right) &= \beta_4(\xi)\,, & \rho_1\left(\frac{1}{\xi}\right) &= \rho_4(\xi)\,,\\
\beta_2\left(\frac{1}{\xi}\right) &= 1-\beta_2(\xi)\,, & \rho_2\left(\frac{1}{\xi}\right) &= \rho_2(\xi)\,,\\
\beta_3\left(\frac{1}{\xi}\right) &= \xi(1-\xi\beta_3(\xi))\,, & \rho_3\left(\frac{1}{\xi}\right) &= \frac{\rho_3(\xi)}{|\xi|^2}\,,\\
\beta_4\left(\frac{1}{\xi}\right) &= \beta_1(\xi)\,, & \rho_4\left(\frac{1}{\xi}\right) &= \rho_1(\xi)\,,\\
\beta_1\left(1-\frac{1}{\xi}\right) &= \beta_2(\xi)-1\,, & \rho_1\left(1-\frac{1}{\xi}\right) &= \rho_2(\xi)\,,\\
\beta_2\left(1-\frac{1}{\xi}\right) &= -\beta_4(\xi)\,, & \rho_2\left(1-\frac{1}{\xi}\right) &= \rho_4(\xi)\,,\\
\beta_3\left(1-\frac{1}{\xi}\right) &= \xi(\xi\beta_3(\xi)-1)\,, & \rho_3\left(1-\frac{1}{\xi}\right) &= \frac{\rho_3(\xi)}{|\xi|^2}\,,\\
\beta_4\left(1-\frac{1}{\xi}\right) &= -\beta_1(\xi)-1\,, & \rho_4\left(1-\frac{1}{\xi}\right) &= \rho_1(\xi)\,.
\end{aligned}
\tag{B.1}
$$

For the benefit of the interested reader we discuss the transformations invoved in computing the second and third integrals in (4.12). Using the variable of integration $\xi' = 1-\xi$ over the region $1-\mathcal{A}$, the second integral involves

$$
I_2 = \int_{1-\mathcal{A}} dx'dy'\ \left(\bar{\partial}'\beta_2(\xi')\partial'(\bar{\xi}'\bar{\beta}_3(\xi')) - \partial'\beta_2(\xi')\bar{\partial}'(\bar{\xi}'\bar{\beta}_3(\xi'))\right). \tag{B.2}
$$

Using the transformation rules

$$
dx'dy' = dxdy, \quad \partial' = -\partial, \quad \bar{\partial}' = -\bar{\partial}\,, \qquad \beta_2(\xi') = -\beta_1(\xi), \quad \beta_3(\xi') = -\beta_3(\xi)\,, \tag{B.3}
$$

we find that I_2 can be written as the following integral over $\mathcal{A}$:

$$
I_2 = \int_{\mathcal{A}} dxdy\ \left(\bar{\partial}\beta_1\partial((1-\bar{\xi})\bar{\beta}_3) - \partial\beta_1\bar{\partial}((1-\bar{\xi})\bar{\beta}_3)\right). \tag{B.4}
$$

In this integral the argument of all β's is ξ. Using the variable of integration $\xi' = 1-1/\xi$ over $1-1/\mathcal{A}$, the third integral in (4.12) is written as

$$
I_3 = \int_{1-\frac{1}{\mathcal{A}}} dx'dy'\ \left(\bar{\partial}'\beta_2(\xi')\partial'(\bar{\xi}'\bar{\beta}_3(\xi')) - \partial'\beta_2(\xi')\bar{\partial}'(\bar{\xi}'\bar{\beta}_3(\xi'))\right). \tag{B.5}
$$

This time we use the following transformation rules

$$
dx'dy' = \frac{dxdy}{|\xi|^4}\,, \quad \partial' = \xi^2\partial\,, \quad \bar{\partial}' = \bar{\xi}^{2}, \quad \beta_2(\xi') = -\beta_4(\xi)\,, \quad \beta_3(\xi) = \xi(\xi\beta_3(\xi)-1)\,, \tag{B.6}
$$

and the integral I_3 can now be written as an integral over $\mathcal{A}$:

$$I_3 = \int_{\mathcal{A}} dxdy \, \left(-\bar{\partial}\beta_4 \partial[(\bar{\xi}-1)(\bar{\xi}\bar{\beta}_3 - 1)] + \partial\beta_4 \bar{\partial}[(\bar{\xi}-1)(\bar{\xi}\bar{\beta}_3 - 1)] \right). \tag{B.7}$$

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